

# FACIALLY DUAL COMPLETE (NICE) CONES AND LEXICOGRAPHIC TANGENTS

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**ABSTRACT.** We study the boundary structure of closed convex cones, with a focus on facially dual complete (nice) cones. These cones form a proper subset of facially exposed convex cones, and they behave well in the context of duality theory for convex optimization. Using the well-known and very commonly used concept of tangent cones in nonlinear optimization, we introduce some new notions for exposure of faces of convex sets. Based on these new notions, we obtain some necessary conditions and some sufficient conditions for a cone to be facially dual complete using tangent cones and a new notion of lexicographic tangent cones (these are a family of cones obtained from a recursive application of the tangent cone concept). Lexicographic tangent cones are related to Nesterov's lexicographic derivatives.

## 1. INTRODUCTION

Understanding the facial structure of convex cones as it relates to the dual cones is fundamentally useful in convex optimization and analysis. Let  $K$  be a closed convex cone in a finite dimensional Euclidean space  $\mathbb{E}$ . For a given scalar product  $\langle \cdot, \cdot \rangle$ , the dual cone is

$$K^* := \{s \in \mathbb{E}^* : \langle s, x \rangle \geq 0 \quad \forall x \in K\},$$

where  $\mathbb{E}^*$  denotes the dual space. Let  $C \subseteq \mathbb{E}$  be a closed convex set. A closed convex subset  $F \subseteq C$  is called a *face* of  $C$  if for every  $x \in F$  and every  $y, z \in C$  such that  $x \in (y, z)$ , we have  $y, z \in F$ . The fact that  $F$  is a face of  $C$  is denoted by  $F \trianglelefteq C$ . Observe that the empty set and the set  $C$  are both faces of  $C$ . Just like other partial orders in this paper, if we write  $F \triangleleft C$ , then we mean  $F$  is a face of  $C$  but is not equal to  $C$ . A nonempty face  $F \triangleleft C$  is called *proper*. Note that if  $K$  is a closed convex cone and  $F \trianglelefteq K$ , then  $F$  is a closed convex cone.

We say that a face  $F$  of a closed convex set  $C$  is *exposed* if there exists a supporting hyperplane  $H$  to the set  $C$  such that  $F = C \cap H$ . Many convex sets have unexposed faces, e.g., convex hull of a torus (see Fig. 1). Another example of a convex set with unexposed faces is the convex hull of a closed unit ball and a disjoint point at the intersection of two orthogonal tangents to the ball (see for instance [15] and Fig. 2 here).

A closed convex set is *facially exposed* if every proper face of  $C$  is exposed. *Facial exposedness* is fundamental in understanding the boundary structure of convex sets; it even has consequences in the theory of convex representations [2, 5]. *Symmetric cones* and *homogeneous cones* are facially exposed (see [4, 23, 22]). *Hyperbolicity cones* are facially exposed too [19], and they represent

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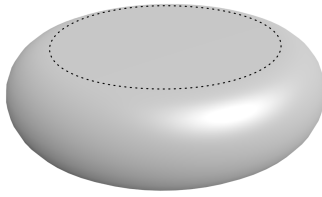


FIGURE 1. Convex hull of a torus is not facially exposed: the dashed line shows the set the extreme points which are not exposed (see [20]).

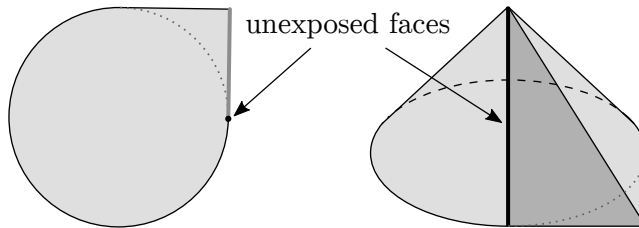


FIGURE 2. An example of a two dimensional set and a three dimensional cone that have an unexposed face.

a powerful and interesting generalization of symmetric cones and homogeneous cones for convex optimization [6, 19] and many other research areas.

Suppose that for a given family of convex optimization problems in conic form, we know that there is at least an optimal solution that is contained in a face  $F$  of  $K$ . We may not have a direct access to the face  $F$ , but perhaps we know the linear span of the face  $F$ :  $\text{span}(F)$ . Then, to compute an optimal solution, we may replace the cone constraint  $x \in K$ , by  $x \in (K \cap \text{span}(F))$ . Now, if we write down the dual problem, the dual cone constraint (for the dual slack variable  $s$ ) becomes (see Proposition 1):

$$s \in (K \cap \text{span}(F))^* = \text{cl} \left( K^* + F^\perp \right)$$

where  $F^\perp := \{s \in \mathbb{E}^* : \langle s, x \rangle = 0 \ \forall x \in F\}$ . Indeed, if  $(K^* + F^\perp)$  happens to be closed, then we can remove the closure operation; otherwise, we would have to deal with this closure operation in some way. Beginning with this observation, we have our first hints for the uses of the concept of *Facially Dual Complete* convex cones. Closed convex cones  $K$  with the property that

$$\left( K^* + F^\perp \right) \text{ is closed for every proper face } F \triangleleft K,$$

are called *Facially Dual Complete (FDC)*. Pataki [14, 15] called such cones *nice*. FDC property is one of the main concepts that we study in this paper. Our interest in FDCness is motivated by many factors:

- FDC property is very important in duality theory. Presence of facial dual completeness makes various facial reduction algorithms behave well, see Borwein and Wolkowicz [1]. Also see Waki and Muramatsu [26] for a variant of facial reduction that does not rely on this property. Currently, the only exact characterization of FDCness is via facial reduction (see Liu and Pataki [11]). For some other recent work related to facial reduction, see [8, 25, 9, 24, 16, 18, 17, 10, 3, 12].
- FDC property is also relevant in the fundamental subject of closedness of the image of a convex set under a linear map. See Pataki [14] and Liu and Pataki [10, 11] and the references therein.

- FDC property comes up in the area of lifted convex representations (see [5]) and in representations of a family of convex cones as a slice of another family of convex cones (see [2]).
- FDC property seems to have a rather mysterious connection to facial exposedness of the underlying cone which is an intriguing and rather beautiful geometric property (see Pataki [15]). Moreover, better understanding of FDC property contributes to our understanding of the boundary structure of convex sets.

Our paper is organized as follows. In Section 3 we recall some notation and some of the known results related to the facial structure of convex cones, then state and prove the necessary and sufficient conditions for facial dual completeness (Theorems 1 and 3). Throughout this process, we introduce some new notions for exposure of faces. In Figure 3 we summarize some of the relationships among various exposure properties. Up to and including 3-dimensions, for convex cones, all of the four properties we listed in Fig. 3 are precisely the same. Starting in 4-dimensions, these four properties identify different sets of convex cones. We are able to illustrate these 4-dimensional convex cones, by taking 3-dimensional slices.

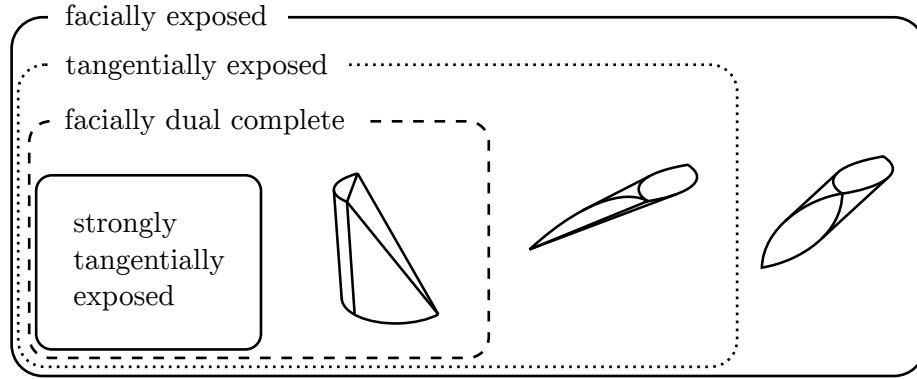


FIGURE 3. Relationships among various notions of facial exposure and FDCness. The graphics represent the examples discussed in this paper.

## 2. PRELIMINARIES

Let  $\mathbb{E}$  denote a finite dimensional Euclidean vector space, and let  $\mathbb{E}^*$  be its dual. Throughout this section by  $K$  we denote a closed convex cone in  $\mathbb{E}$ . We call  $K$  *regular* if  $K$  is pointed (does not contain whole lines), closed, convex and has nonempty interior in  $\mathbb{E}$ . If  $K$  is a regular cone then so is its dual cone  $K^*$ .

Let  $C \subseteq \mathbb{E}$  and  $x \in C$ . Then, the tangent cone for  $C$  at  $x$  is

$$\text{Tangent}(x; C) := \limsup_{t \rightarrow +\infty} t(C - x).$$

The *cone of feasible directions* of  $C$  at  $x$  is

$$\text{Dir}(x; C) := \{d \in \mathbb{E} : (x + \epsilon d) \in C \text{ for some } \epsilon > 0\}.$$

Note that

$$\text{Tangent}(x; C) = \text{cl Dir}(x; C).$$

The direction  $s \in \mathbb{E}^*$  is said to be *normal* to a closed convex set  $C$  at a point  $x$  if

$$\langle s, y - x \rangle \leq 0 \quad \forall y \in C.$$

The set of all such directions is called the *normal cone at  $x$  to  $C$* , denoted by  $\text{Normal}(x; C)$ .

In addition to the notion of dual cone, we also use the closely related concept of *polar* of a set. For a subset  $C$  of  $\mathbb{E}$ , the *polar* of  $C$  is

$$C^\circ := \{s \in \mathbb{E}^* : \langle s, x \rangle \leq 1 \quad \forall x \in C\}.$$

Note that for cones the notions of dual cone and polar are equivalent. For example, for every convex set  $C$  and for every  $x \in C$ , we have

$$\text{Normal}(x; C) = [\text{Tangent}(x; C)]^\circ \quad \text{and} \quad \text{Tangent}(x; C) = -[\text{Normal}(x; C)]^*.$$

The following fact is used many times in this paper.

**Proposition 1.** *For every pair of closed convex cones  $K_1$  and  $K_2$  in  $\mathbb{E}$ , we have*

$$(K_1 \cap K_2)^* = \text{cl}(K_1^* + K_2^*).$$

*If the relative interiors of  $K_1$  and  $K_2$  have nonempty intersection, then  $K_1^* + K_2^*$  is a closed set and therefore the closure operation can be omitted.*

*Proof.* See Corollary 16.4.2 in Rockafellar [20] and Remark 5.3.1. in [7].  $\square$

On many occasions we will be dealing with sets in  $\mathbb{E}$  that are not full-dimensional. When we refer to the dual cones of such sets, the domain over which the dual cone is defined matters. Let  $F \subset \mathbb{E}$ . Then we may consider the dual cone of  $F$  with respect to any Euclidean space  $L$  such that  $\text{span}(F) \subseteq L \subseteq \mathbb{E}$ . We denote by  $F|_L^*$  the dual cone of  $F$  in  $\mathbb{E}^*/L^\perp$ ; i.e.,

$$F|_L^* := \left\{ s \in \mathbb{E}^*/L^\perp : \langle s, x \rangle \geq 0 \quad \forall x \in F \right\}.$$

Next, we define the projection map in the dual space. For  $C \subseteq \mathbb{E}^*$ ,

$$\Pi_{\mathbb{E}^*/L^\perp}(C) := \{[v] : v \in C\},$$

where  $[v]$  is the equivalence class of  $v \in \mathbb{E}^*$  with respect to  $L^\perp$ . In  $\mathbb{E}^* = \mathbb{R}^n$  this represents the orthogonal projection of  $C$  onto the linear subspace  $L$ . Using Proposition 1, we can easily derive

$$F|_L^* = \Pi_{\mathbb{E}^*/L^\perp}(F^*),$$

where  $F^*$  is the dual cone of  $F$  with respect to  $\mathbb{E}$ .

Let  $C$  be a closed convex set and let  $S$  be a nonempty subset of  $C$ . We define the *minimal face* of  $C$  containing  $S$  as follows:

$$\text{face}(S; C) := \bigcap \{F : F \trianglelefteq C, S \subseteq F\}.$$

The following facts are elementary (and a few are well-known), we present all but one without proof. For  $u \in \mathbb{E}^*$ , we denote

$$u^\perp := \{x \in E : \langle u, x \rangle = 0\}.$$

**Proposition 2** (Properties of faces). *Let  $C$  be a closed convex set in a finite dimensional Euclidean space  $\mathbb{E}$ . Then the following properties are true:*

- (i) *face of a face of  $C$  is a face of  $C$  (i.e.,  $G \trianglelefteq F \trianglelefteq C$  implies  $G \trianglelefteq C$ );*
- (ii) *for every  $x \in C$  and every  $u \in \text{Normal}(x; C)$  the set  $\text{Tangent}(x; F) \cap u^\perp$  is a face of  $\text{Tangent}(x; F)$ ;*
- (iii) *for every  $S \subseteq C$ , we have  $\text{relint}(\text{conv } S) \cap \text{relint}(\text{face}(S; C)) \neq \emptyset$ .*

**Proposition 3.** *Let  $K$  be a closed convex cone. Then for every pair  $(u, x)$  with  $u \in K^*$  and  $x \in (K \cap u^\perp)$ , we have  $u \in [\text{Tangent}(x; F)]^*$ .*

*Proof.* Since  $u$  defines a supporting hyperplane to  $F$  at  $x$ , this supporting hyperplane is also a supporting hyperplane for the tangent cone, and hence  $u \in [\text{Tangent}(x; F)]^*$ .  $\square$

**Proposition 4.** *A closed convex cone  $K \subseteq \mathbb{R}^n$  is FDC if and only if for every face  $F \triangleleft K$*

$$F|_L^* = \Pi_{\mathbb{R}^*/L^\perp}(K^*),$$

where  $L := \text{span } F$ .

### 3. FACIALLY DUAL COMPLETE CONES AND TANGENTIAL EXPOSURE

We say that a closed convex set  $C$  in a finite dimensional Euclidean space  $\mathbb{E}$  has *tangential exposure* property if

$$(1) \quad \text{Tangent}(x; K) \cap \text{span } F = \text{Tangent}(x; F) \quad \forall F \triangleleft K, \forall x \in F.$$

Tangential exposure is a stronger property than facial exposure. We discuss the relation between these two notions and provide illustrative examples later in this section. We next prove Theorem 1 that gives a necessary condition for the FDC property, establishing that every FDC cone is tangentially exposed.

#### 3.1. Proof of the necessary condition.

**Theorem 1.** *If a closed convex cone  $K$  is facially dual complete, then for every  $F \triangleleft K$  and every  $x \in F$ , we have*

$$(2) \quad \text{Tangent}(x; K) \cap \text{span } F = \text{Tangent}(x; F).$$

*Proof.* Since  $\text{Tangent}(x; F)$  is a subset of both  $\text{Tangent}(x; K)$  and  $\text{span } F$ , the inclusion

$$\text{Tangent}(x; K) \cap \text{span } F \supseteq \text{Tangent}(x; F)$$

follows. For the other inclusion, for the sake of reaching a contradiction, assume the contrary:  $K$  is facially dual complete, but there exist  $F \triangleleft K$  and  $x \in F$  such that (2) does not hold. Then, there exists  $g \in \text{Tangent}(x; K) \cap \text{span } F$  such that  $g \notin \text{Tangent}(x; F)$ . Without loss of generality, we may assume  $\|g\| = 1$ . Since  $g \in \text{span } F =: L$ , applying the hyperplane separation theorem to  $g$  and  $\text{Tangent}(x; F)$ , in the space of  $\text{span } F$ , we deduce that there exists  $p \in \text{Normal}(x; F)|_L^*$  such that  $\langle p, g \rangle > 0$ .

Since  $F$  is a cone, we have  $\text{Normal}(x; F) \subseteq \text{Normal}(0; F) = -F^*$ , hence,  $p \in -F^*$ . Since  $K$  is facially dual complete, by Remark 1 in [15] we have  $F^* = K^* + F^\perp$ ; hence, there exist  $y \in -K^*$  and  $z \in F^\perp$  such that  $y = p - z$ . Since  $g \in \text{span } F$  and  $z \in F^\perp$ , we have

$$\langle y, g \rangle = \langle p - z, g \rangle = \langle p, g \rangle > 0.$$

Since  $g \in \text{Tangent}(x; K)$ , there exists a sequence  $\{s_k\}$ , such that  $s_k \in K$  and

$$\lim_{k \rightarrow \infty} \frac{s_k - x}{\|s_k - x\|} = g.$$

Therefore,

$$\lim_{k \rightarrow \infty} \frac{\langle s_k - x, y \rangle}{\|s_k - x\|} = \langle g, y \rangle > 0,$$

and there exists  $k$  large enough such that

$$\langle s_k - x, y \rangle > 0.$$

Now observe that since  $F$  is a cone, and  $x \in F$ , we also have  $\frac{1}{2}x \in F$  and  $\frac{3}{2}x \in F$ , hence, by the convexity of  $F$ ,

$$-\frac{1}{2}x, \frac{1}{2}x \in \text{Tangent}(x; F).$$

Since  $p \in \text{Normal}(x; F)$ , this yields  $\langle p, x \rangle = 0$ . Then  $\langle x, y \rangle = \langle x, p \rangle - \langle x, z \rangle = 0$ , and we have

$$0 < \langle s_k - x, y \rangle = \langle s_k, y \rangle.$$

However, this is impossible, as  $s_k \in K$ ,  $y \in -K^*$ , and hence  $\langle s_k, y \rangle \leq 0$ . Therefore, our assumption is not true, and by the arbitrariness of  $F$  and  $x$  we have shown that (2) holds for all  $F \triangleleft K$  and all  $x \in F$ .  $\square$

There are regular cones which are facially exposed, not FDC and not tangentially exposed. The example from Roshchina [21] satisfies these properties. See, Figure 4. Nevertheless, there are facially exposed regular cones that are also tangentially exposed, but not FDC. We can prove this by modifying the example from [21].

**Example 1.** We revisit the example from [21]. The closed convex cone  $K \subset \mathbb{R}^4$  is a standard homogenization  $K = \text{cone}\{C \times \{1\}\}$  of a compact convex set  $C \subset \mathbb{R}^3$  whose construction and Mathematica rendering are shown in Fig. 4. The set  $C$  is a nonsingular affine transformation of

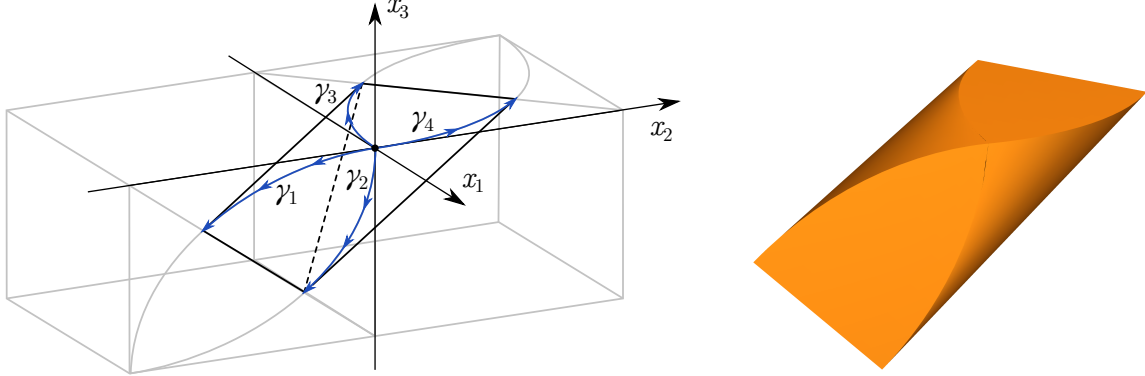


FIGURE 4. A slice of a closed convex cone that is facially exposed but not FDC. Notice that this set is not strongly facially exposed.

the convex hull of four curves. In particular, it is  $\text{conv}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ , where

$$\begin{aligned} \gamma_1(t) &:= (0, -\sin t, \cos t - 1), & \gamma_2(t) &:= (0, \cos t - 1, -\sin t), \\ \gamma_3(t) &:= (-\sin t, 1 - \cos t, 0), & \gamma_4(t) &:= (\cos t - 1, \sin t, 0), \end{aligned}$$

and  $t \in [0, \pi/4]$ . It is not difficult to observe that if  $C$  fails the tangential exposure property, then its homogenization  $K$  does as well. The failure of tangential exposure for the set  $C$  is evident from considering tangents to the face  $F = \text{conv}\{\gamma_3, \gamma_4\}$  and  $C$  at the point  $(0, 0, 0)$ . Indeed, it is clear that  $g := (0, -1, 0) \in \text{Tangent}(x; K)$  since

$$(0, -1, 0) = \limsup_{t \rightarrow \infty} t\gamma_1(t^{-1}) = \lim_{s \downarrow 0} \frac{(0, -\sin s, \cos s - 1)}{s}.$$

On the other hand,

$$\langle g, \gamma_3(t) \rangle = \cos t - 1 \leq 0, \quad \langle g, \gamma_4(t) \rangle = -\sin t \leq 0 \quad \forall t \in [0, \pi/4],$$

hence  $g$  is separated strictly from  $\text{Tangent}(x; F)$ . This is illustrated geometrically in Fig. 5.

**Example 2.** We construct a modified example of a closed convex cone that is facially *and* tangentially exposed, but is not facially dual complete. This cone is a homogenization of the three-dimensional set  $C$  that is a convex hull of two curves, one is a piece of a parabola, and the other one is a twisted cubic (see Fig. 6). So, we have  $K = \text{cone}\{C \times \{1\}\}$ ,  $C = \text{conv}\{\gamma_1, \gamma_2\}$ , where

$$\gamma_1(s) = (-s, -s^2, -s^3), \quad s \in [0, 1] \quad \text{and} \quad \gamma_2(t) = (-t, t^2, 0), \quad t \in [0, 1/3(2 + \sqrt{7})].$$

It is a technical exercise to show that the cone  $K$  (or equivalently the set  $C$ ) is tangentially exposed, but not FDC. We leave the detailed algebraic computations, as well as the proof that the set is not FDC, to the appendix.

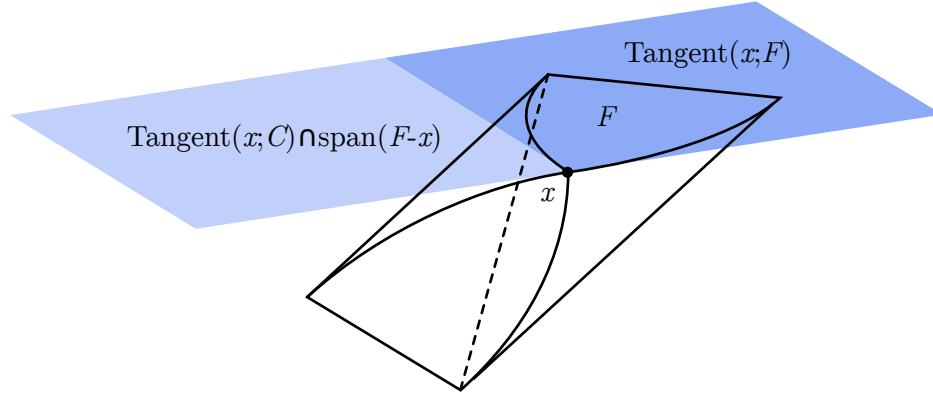


FIGURE 5. Failure of tangential exposure

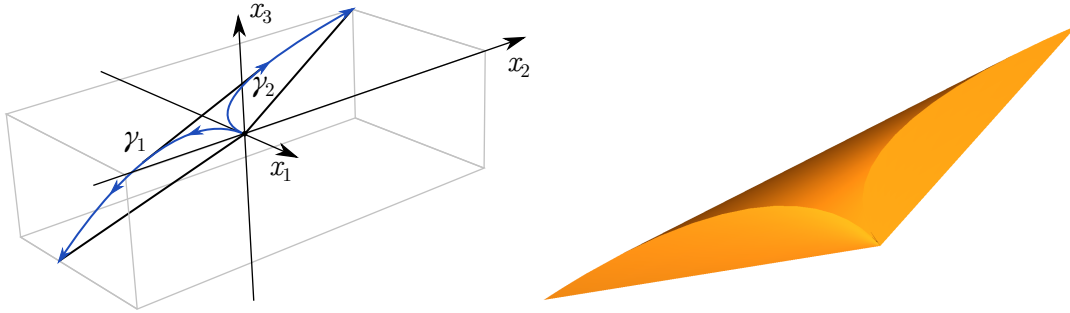


FIGURE 6. A slice of a closed convex cone that is tangentially exposed but not facially dual complete.

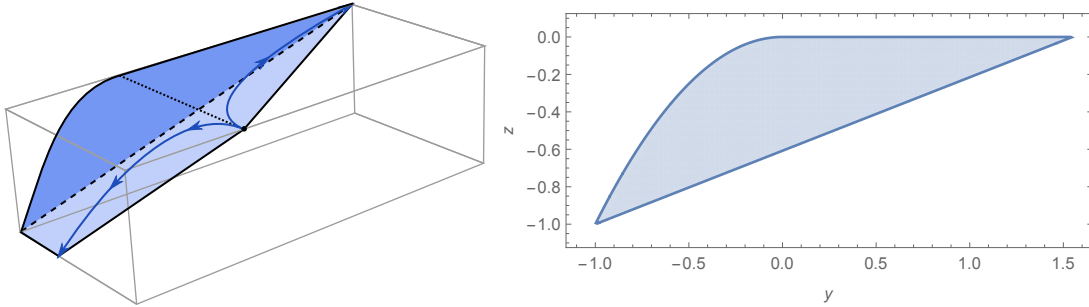


FIGURE 7. Failure of tangential exposure for the tangent cone

**3.2. Lexicographic tangent cones.** The last example leads us to the next idea. The above regular cone is facially exposed and tangentially exposed, but it is not FDC. Also, its tangent cone to  $C$  at  $x = (0, 0, 0)$  is not tangentially exposed itself. This is intuitively clear from Fig. 7, where the dotted line in the left-hand-side graphic shows the set of points for which the strong tangential exposure fails with respect to the adjacent flat face, and the right-hand-side plot shows the slice of this second order tangent cone. So, we consider a stronger property defined by enforcing tangential exposure condition (2) recursively on all tangent cones. For example, a second-order

tangent cone for  $C$  at  $x \in C$  and  $v \in \text{Tangent}(x; C)$  is:

$$\begin{aligned} \text{Tangent}[v; \text{Tangent}(x; C)] &= \limsup_{t_2 \rightarrow +\infty} t_2 [T(x; C) - v] \\ &= \limsup_{t_2 \rightarrow +\infty} t_2 \left\{ \left[ \limsup_{t_1 \rightarrow +\infty} t_1 (C - x) \right] - v \right\}. \end{aligned}$$

We may recursively apply this construction to generate  $k$ th-order tangent cones for every non-negative integer  $k$ . This geometric notion is a geometric counterpart of Nesterov's *lexicographic derivatives* (see [13] for this analytic notion, and the references therein). Any tangent cone obtained as a result of the above recursive procedure (of any order) is called a *lexicographic tangent cone* of  $C$ . We say that a closed convex set is *strongly tangentially exposed* if it is tangentially exposed along with all of its lexicographic tangent cones.

Next, we investigate some fundamental properties of the family of lexicographic tangent cones of closed convex sets. Observe that for  $u, v \in C$  such that  $\text{face}(u; C) = \text{face}(v; C) =: F$ , we have

$$\text{Tangent}(u; C) = \text{Tangent}(v; C) =: \text{Tangent}(F; C).$$

That is,  $\text{Tangent}(F; C)$  denotes the tangent cone for  $C$  at any  $x \in \text{relint } F$  for  $F \trianglelefteq C$ . Thus, the cardinality of distinct tangent cones of  $C$  is bounded by the cardinality of the set of faces of  $C$ . With this notation, our Theorem 1 can be restated as:

Let  $K$  be a regular cone that is FDC. Then for every pair of faces  $F, G$  such that  $G \triangleleft F \trianglelefteq K$ , we have

$$\text{Tangent}(G; K) \cap \text{span } F = \text{Tangent}(G; F).$$

Let

$$\mathcal{T} : \text{subsets of convex sets in } \mathbb{E} \rightarrow \text{subsets of convex cones in } \mathbb{E},$$

defined by

$$\mathcal{T}(\mathcal{K}) := \text{the set of all tangent cones of convex sets in } \mathcal{K}.$$

We further define  $\mathcal{T}^0(\mathcal{K}) := \mathcal{K}$  and for every positive integer  $k$ ,  $\mathcal{T}^k(\mathcal{K}) := \mathcal{T}[\mathcal{T}^{k-1}(\mathcal{K})]$ . Note that, if for some family of convex sets  $\mathcal{K}$ , we have  $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ , then

$$\mathcal{T}^k(\mathcal{K}) = \mathcal{K}, \quad \text{for every nonnegative integer } k.$$

Let  $C$  be a closed convex set. Then, *tangential depth* of  $C$  is the smallest nonnegative integer  $k$  such that  $\mathcal{T}^{k+1}(C) = \mathcal{T}^k(C)$ . (When  $\mathcal{K}$  is a singleton  $C$ , we write  $\mathcal{T}(C)$  instead of  $\mathcal{T}(\{C\})$ .) Next, we prove that the tangential depth of every regular cone is bounded by its dimension.

**Theorem 2.** *For every closed convex cone  $K$  in  $\mathbb{E}$ , the tangential depth of  $K$  is at most the dimension of  $K$ .*

*Proof.* For every  $F \triangleleft K$ ,  $\text{Tangent}(F; K)$  has a lineality space of dimension at least  $\dim(F)$ . If  $\dim(F) = 0$ , then  $\text{Tangent}(F; K) = K$ . Now, let  $T \in \mathcal{T}^k(K)$  for some nonnegative integer  $k$ . Then  $T = \bar{T} + L$ , where  $L$  is the lineality space of  $T$  and  $\bar{T}$  is a pointed closed convex cone such that  $\dim(\bar{T}) = \dim(T) - \dim(L)$ . Let  $\ell := \dim(L)$ . Then, every cone in  $\mathcal{T}(T)$  except for  $T$ , has a lineality space of dimension at least  $(\ell + 1)$ . Therefore, with  $n := \dim(K)$ , if  $\mathcal{T}^n(K) \neq \mathcal{T}^{(n-1)}(K)$ , the only tangent cone in  $\mathcal{T}^n(K)$  that was not in  $\mathcal{T}^{(n-1)}(K)$  would have to be the trivial cone of  $n$ -dimensional Euclidean space. Therefore,  $\mathcal{T}^{(n+1)}(K) = \mathcal{T}^n(K)$  and by the observation before the theorem, the tangential depth of  $K$  is at most  $n$ .  $\square$

Therefore, a regular cone  $K$  is strongly tangentially exposed iff every cone in the set  $\mathcal{T}^n(K)$  is tangentially exposed, where  $n := \dim(K)$ . Our next goal is to prove that strongly tangentially exposed closed convex cones are FDC.



**3.3. Proof of the sufficient condition.** We use several technical claims in the proof. The next proposition immediately follows from the above definitions.

**Proposition 5.** *Tangent cones inherit strong tangential exposure property from the original object. That is, if  $C$  is strongly tangentially exposed, then every  $T \in \mathcal{T}^k(C)$  is strongly tangentially exposed for every nonnegative integer  $k$ .*

**Proposition 6.** *Let  $K$  be a regular cone, and let  $F \triangleleft K$  be an exposed face of  $K$ ,  $L := \text{span } F$ . Then for every  $u \in F|_L^*$  such that  $u$  exposes  $\{0\}$  as a face of  $F$ , there exists  $g \in K^*$  such that  $u = \Pi_{\mathbb{E}^*/L^\perp} g$ .*

*Proof.* Let  $K, F$ , and  $L$  be as above. Choose an arbitrary  $u \in F|_L^*$  such that  $\langle u, x \rangle > 0, \forall x \in F \setminus \{0\}$ . Without loss of generality, we may assume  $\|u\| = 1$ . Since  $F$  is an exposed proper face of  $K$ , there exists  $s \in K^*$  such that

$$\langle s, x \rangle \begin{cases} = 0, & \text{if } x \in F; \\ > 0, & \text{if } x \in K \setminus F. \end{cases}$$

Let  $g_\alpha := u + \alpha s, \alpha \in \mathbb{R}$ . If there exists  $\alpha$  such that  $g_\alpha \in K^*$ , then we are done. So, we may assume that for every  $\alpha \in \mathbb{R}$ , there exists  $x_\alpha \in K$  such that

$$0 > \langle g_\alpha, x_\alpha \rangle = \langle u, x_\alpha \rangle + \alpha \langle s, x_\alpha \rangle$$

Since  $K$  is a cone, we can choose  $x_\alpha$  to be unit norm. Now, as  $\alpha \rightarrow +\infty$ , the sequence  $x_\alpha$  must have a convergent subsequence with limit  $\bar{x} \in K$  which also has norm 1. If  $\langle s, \bar{x} \rangle > 0$ , then using

$$-1 \leq -\|u\|\|x_\alpha\| \leq \langle u, x_\alpha \rangle < -\alpha \langle s, x_\alpha \rangle$$

and taking limits as  $\alpha \rightarrow +\infty$  along the subsequence of  $\{x_\alpha\}$  converging to  $\bar{x}$ , we reach a contradiction. Hence, we may assume  $\langle s, \bar{x} \rangle = 0$ , i.e.,  $\bar{x} \in F$ . Applying the above limit argument with this new information, we conclude  $\langle u, \bar{x} \rangle \leq 0$ . Thus, by our choice of  $u$ ,  $\bar{x} = 0$ , again leading to a contradiction. Therefore, there exists  $\alpha$  such that  $g_\alpha \in K^*$ , and we are done.  $\square$

Next, we observe that FDCness and strong tangential exposedness are not affected by addition or removal of subspaces.

**Proposition 7.** *Let  $K = C + L$ , where  $L$  is a maximal linear subspace and  $C$  is a closed convex cone such that  $C \cap L = \{0\}$ . Then the following statements are true.*

- (i) *The cone  $K$  is strongly tangentially exposed if and only if  $C$  is;*
- (ii) *The cone  $K$  is FDC if and only if  $C$  is.*

*Proof.* Proof of (i) is contained in the proof of Theorem 2. Proof of (ii) follows from definitions and fundamental properties.  $\square$

Now, we are ready to prove our sufficient condition for FDCness.

**Theorem 3** (Sufficient condition). *If a closed convex cone  $K \subseteq \mathbb{E}$  is strongly tangentially exposed, then it is facially dual complete.*

*Proof.* We will prove the statement by induction in the dimension of the underlying space  $n := \dim(\mathbb{E})$ . Observe that for  $n = 1$  the statement is trivial: all three possible, at most one-dimensional, nonempty, closed convex cones are both strongly tangentially exposed and facially dual complete.

Assume now that every closed convex cone of dimension at most  $(n - 1)$  that is strongly tangentially exposed is also FDC. We will prove the statement for  $n$ -dimensional closed convex cones. Let  $\mathbb{E}$  be an  $n$ -dimensional space and let  $K \subseteq \mathbb{E}$  be a strongly tangentially exposed closed convex cone. To prove that  $K$  is FDC, by Proposition 4 it suffices to show that for all  $F \triangleleft K$ , with  $L := \text{span } F$ , for every  $u \in F|_L^*$ , we have  $u \in \Pi_{\mathbb{E}^*/L^\perp} K^*$ .

Choose an arbitrary  $F \triangleleft K$  and  $u \in F|_L^*$ . Let

$$E := \{x \in F : \langle u, x \rangle = 0\}.$$

Observe that  $E \triangleleft F \triangleleft K$ , since  $u$  defines a supporting hyperplane to  $F$  at origin, and any sub-face of a face is also a face (see Proposition 2), if  $E = \{0\}$ , the result follows from Proposition 6. Otherwise  $\dim E \geq 1$ . Let  $x \in \text{relint } E$  and consider  $\text{Tangent}(x; K)$  and  $\text{Tangent}(x; F)$ . Observe that  $\text{span } E \subset \text{Tangent}(x; F) \subset \text{Tangent}(x; K)$ , so that our cones decompose into a direct sum:

$$\text{Tangent}(x; K) = C + \text{span } E,$$

where  $C \cap \text{span } E = \{0\}$ . Notice that since  $\dim E \geq 1$ , we have  $\dim C \leq n - 1$ .

By Proposition 5 the tangent  $\text{Tangent}(x; K)$  inherits strong tangential exposedness property from  $K$ . Applying Proposition 7 (i) to  $\text{Tangent}(x; K)$  and  $C$ , we deduce that  $C$  is strongly tangentially exposed as well, and since the dimension of  $C$  is less than  $n$ , it is FDC by the induction hypothesis. Applying Proposition 7 (ii) to  $\text{Tangent}(x; K)$  and  $C$ , we deduce that  $\text{Tangent}(x; K)$  is facially dual complete.

By our choice of  $x$  and Proposition 3, letting  $L := \text{span}(\text{Tangent}(x; F))$ , we have

$$(3) \quad u \in (\text{Tangent}(x; F))|_L^*.$$

If  $\text{Tangent}(x; F)$  is a face of  $\text{Tangent}(x; K)$ , then from the FDCness of  $\text{Tangent}(x; K)$  there exists  $g \in (\text{Tangent}(x; K))^* \subset K^*$  such that with  $W := \text{span } \text{Tangent}(x; F)$ ,  $u = \Pi_{W^*} g = \Pi_{\mathbb{E}^*/L^\perp} g$ , and we are done.

If  $\text{Tangent}(x; F)$  is not a face of  $\text{Tangent}(x; K)$ , then consider the minimal face  $G \triangleleft \text{Tangent}(x; K)$  that contains  $\text{Tangent}(x; F)$ . By the property of minimal faces in Proposition 2 we have

$$\text{relint } [\text{Tangent}(x; F)] \cap \text{relint } G \neq \emptyset,$$

and therefore

$$\{\text{span } [\text{Tangent}(x; F)]\} \cap \text{relint } G \neq \emptyset.$$

Applying Proposition 1 to  $[\text{span } \text{Tangent}(x; F)]$  and  $G$ , we have

$$(4) \quad \{[\text{span } \text{Tangent}(x; F)] \cap G\}^* = G^* + [\text{Tangent}(x; F)]^\perp.$$

From the strong tangential exposure assumption we have

$$\text{Tangent}(x; F) = \text{Tangent}(x; K) \cap \text{span } \text{Tangent}(x; F),$$

and since  $\text{Tangent}(x; F) \subseteq G \subseteq \text{Tangent}(x; K)$ , this yields

$$(5) \quad [\text{span } \text{Tangent}(x; F)] \cap G = \text{Tangent}(x; F).$$

From (4) and (5) we have:

$$[\text{Tangent}(x; F)]^* = G^* + [\text{Tangent}(x; F)]^\perp.$$

From (3), with  $U := \text{span } G$ , we have

$$u \in [\text{Tangent}(x; F)]^* \cap [\text{span } \text{Tangent}(x; F)] \subseteq [\text{Tangent}(x; F)]|_U^*,$$

and so there exists  $g \in G^*$  that projects onto  $u$  (under the projection onto  $[\text{span } \text{Tangent}(x; F)]$ ). Since  $G$  is a face of  $\text{Tangent}(x; K)$ , and  $\text{Tangent}(x; K)$  is FDC, we can now find a point  $g'$  in  $(\text{Tangent}(x; K))^* \subset K^*$  that projects onto  $\mathbb{E}^*/U^\perp$  as  $g$ . It is not difficult to observe that its projection onto  $\text{span } F = \text{span } \text{Tangent}(x; F)$  is then  $u$ .  $\square$

The sufficient condition for FDCness is not necessary, as is evident from the next example.

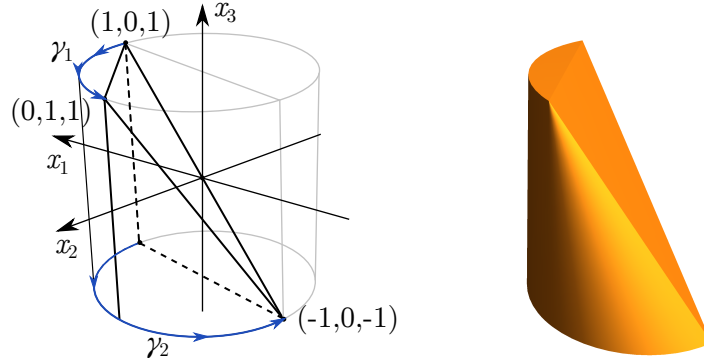


FIGURE 8. Facially exposed set may have a tangent that is not facially exposed

**Example 3.** Let  $K = \text{cone}\{C \times \{1\}\} \subset \mathbb{R}^4$ , where  $C \subset \mathbb{R}^3$  is a closed convex set,  $C := \text{conv}\{\gamma_1, \gamma_2\}$ ,

$$\gamma_1(t) = (\cos t, \sin t, 1), \quad t \in [0, \pi/2], \quad \gamma_2(t) = (\cos t, \sin t, -1) \quad t \in [0, \pi].$$

The set  $C$  is shown in Fig. 8. Observe that the set  $C$  is tangentially (and facially) exposed. However, strong tangential exposure fails for this set. In particular,  $\text{Tangent}(\bar{x}, C)$ , where  $\bar{x} = (0, 1, 1)$  is not facially exposed (see its Mathematica rendering in the first image of Fig. 9), and

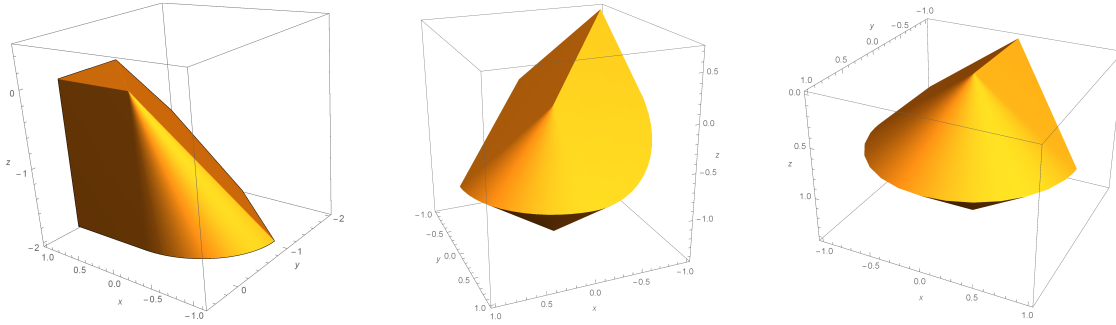


FIGURE 9. Tangent cone that is not facially exposed and the two closed convex sets whose conic hulls represent the projections of the dual cones on the relevant subspaces.

hence it is not tangentially exposed either. At the same time this cone is facially dual complete. In this case we only need to check the identity  $\Pi_{\mathbb{E}^*/F^\perp}(F^\perp + K^*) = F|_L^*$  for the faces of  $K$  that correspond to the top and bottom faces of  $C$ , and for both cases the relevant projections are the conic hulls of three dimensional sets shown in the last two images in Fig. 9. We provide all relevant technical computations in the appendix.

#### 4. CONCLUSION

We provided tighter, geometric, primal characterizations of facial dual completeness of regular convex cones via tangential exposure property and strong tangential exposure property. Our results provide geometric tools for checking FDCness. As a by-product of our approach, we have introduced some new notions of exposure for faces of closed convex sets:

- (i) tangentially exposed convex sets
- (ii) convex sets with facially exposed tangent cones
- (iii) convex sets with every lexicographic tangent cone facially exposed
- (iv) strongly tangentially exposed convex sets.

We can also apply these notions to the polars of convex sets. Also, we can ask characterizations of closed convex sets  $C$  such that  $C$  and  $C^\circ$  have a specific property (or a specific pair of the properties) from the above list.

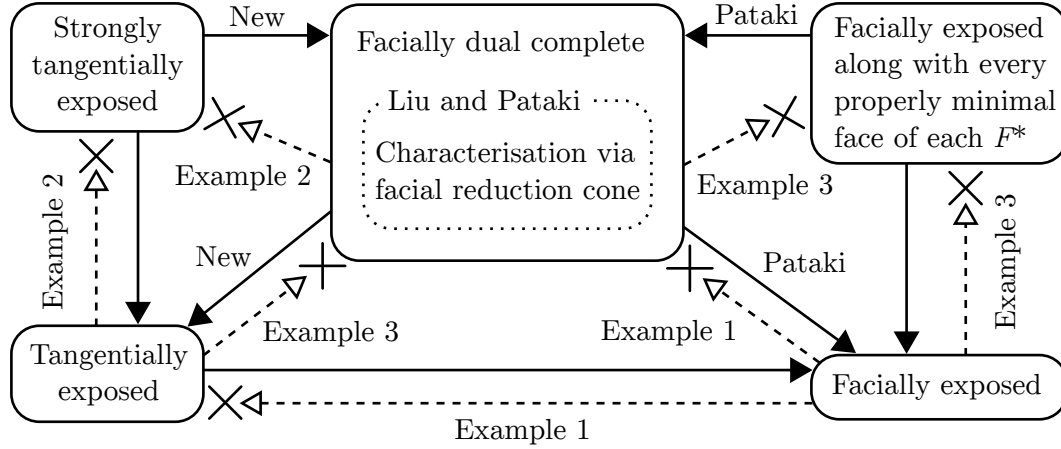


FIGURE 10. A schematic summary of results

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## APPENDIX

Throughout the appendix, as we are working with concrete examples in  $\mathbb{R}^3$  and  $\mathbb{R}^4$ , we identify primal and dual spaces.

**Technical computations for Example 2.** We verify algebraically that the cone  $K$  in Example 2 is tangentially exposed but not strongly tangentially exposed, and that  $K$  is FDC.

Observe that facial and tangential exposedness can be checked directly for the slice  $C$  rather than the cone  $K$  itself.

*Faces of  $C$ .* We identify all possible faces of the set  $C$ , and hence all but trivial faces of the cone  $K$ . We first show that the parametric families

$$F_{11}(s) = [0, \gamma_1(s)], \quad s \in (0, 1], \quad F_{22}(s) = [\gamma_1(s), \gamma_2(\varphi(s))], \quad s \in (0, 1],$$

where  $\varphi(s) = 1/3(2 + \sqrt{7})s$ , are one-dimensional faces of  $C$ , and

$$F_1 = \text{conv}\{0, \gamma_1(1), \gamma_2(\varphi(1))\}, \quad F_2 = \text{conv}\{\gamma_2\}$$

are two dimensional faces of  $C$ .

It immediately follows from the above listing of faces that  $\gamma_1 \cup \gamma_2 \subseteq \text{ext } C$ , since all points in  $\gamma_1 \cup \gamma_2$  are subfaces of higher dimensional faces listed above. At the same time, the only possible extreme points of  $C$  belong to  $\gamma_1 \cup \gamma_2$ . Indeed, if there is an extreme point of  $C$  that does not belong to  $S = \gamma_1 \cup \gamma_2$ , we can represent it as a convex combination of points from  $S$  with nonzero coefficients, and hence obtain a line segment that is a subset of the same face, which contradict the definition of a face.

We next prove that  $F_1, F_2, F_{11}(s)$  and  $F_{12}(s)$  are exposed faces of  $C$ , and then demonstrate that there are no faces of  $C$  other than the ones that we have just listed.

*The sets  $F_1$  and  $F_2$  are faces of  $C$ .* To show that  $F_1 = \text{conv}\{0, \gamma_1(1), \gamma_2(\varphi(1))\}$  is a face of  $C$  we prove that the point

$$w := \left( 11 + 4\sqrt{7}, 3 \left( 2 + \sqrt{7} \right), -17 - 7\sqrt{7} \right)$$

is its (outer) exposing normal. It is not difficult to observe that

$$\langle w, \gamma_1(1) \rangle = 0, \quad \langle w, \gamma_2(\varphi(1)) \rangle = 0 \quad \langle w, 0 \rangle = 0.$$

Moreover, for all  $s \in (0, 1)$  we have

$$\langle w, \gamma_1(s) \rangle = s(s-1)(11 + 4\sqrt{7} + 17s + 7\sqrt{7}s) < 0,$$

$$\langle w, \gamma_2(\varphi(s)) \rangle = \frac{s(s-1)(2 + \sqrt{7})(11 + 4\sqrt{7})}{3} < 0,$$

therefore  $F_1$  is an exposed face of  $C$ .

For the set  $F_2 = \text{conv}\{0, \gamma_2\}$  and the point  $w = (0, 0, 1)$  observe that

$$\langle \gamma_2, w \rangle = 0, \quad \langle \gamma_1(s), w \rangle = \begin{cases} 0, & s = 0, \\ -s^3 < 0, & s > 0 \end{cases},$$

hence,  $F_2$  is an exposed face of  $C$  as well.

*One dimensional faces of  $C$ .* For the family  $F_{11}(s)$  connecting the point  $\gamma_1(s)$  with its endpoint  $\gamma_1(0) = 0$  we have the normal

$$w(s) = (s^2, -2s, 1).$$

Indeed,

$$\langle \gamma_1(u), w \rangle = -u(u-s)^2,$$

which is strictly less than zero for  $u \neq s$ ,  $s > 0$  and equals zero for  $u = s$ , and

$$\langle \gamma_2(t), w(s) \rangle = -s^2t - 2st = -st(2t + s) < 0 \quad \forall t \in (0, \varphi(1)].$$

For the family  $F_{12}(s)$  joining the two curves we have the exposing normal

$$w(s) = \left( 2 \left( -\sqrt{7} - 1 \right) s^2, \left( \sqrt{7} - 5 \right) s, 4 \right).$$

To check that this normal indeed gives us the faces in  $F_{12}(s)$ , we calculate

$$f(u) := \langle \gamma_1(u), w \rangle = 2 \left( \sqrt{7} + 1 \right) s^2 u - \left( \sqrt{7} - 5 \right) s u^2 - 4u^3.$$

Solving

$$f'(u) = 2 \left( \sqrt{7} + 1 \right) s^2 - 2 \left( \sqrt{7} - 5 \right) s u - 12u^2 = 0$$

produces two solutions,

$$u = s \quad \text{and} \quad u = -1/6(1 + \sqrt{7})s.$$

where the second one is not on  $[0, 1]$ , so there is only one possible maximum on  $[0, 1]$ . Checking the second derivative

$$f''(s) = -2 \left( \sqrt{7} - 5 \right) s - 24s = (14 - 2\sqrt{7}) < 0,$$

we deduce that at  $u = s$  we have a strict local maximum. There are no other extreme points on  $[0, 1]$ , hence the whole curve  $\gamma_1$  lies on the nonpositive side of the hyperplane.

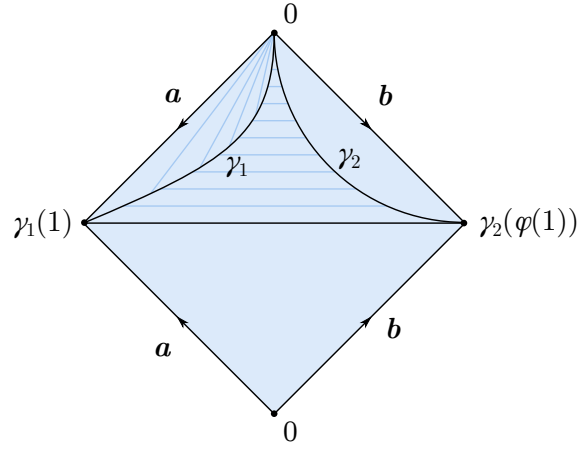
For the curve  $\gamma_2$  we have

$$\langle \gamma_2(\varphi(u)), w(s) \rangle = \left( 3 + \sqrt{7} \right) (2s^2u - su^2).$$

Calculating the derivative for  $f(u) = 2s^2u - su^2$  we have

$$f'(u) = 2s^2 - 2su = 2s(s - u),$$

and we have one root  $u = s$ . Checking the second derivative  $f''(u) = -2s$ , we conclude that this is a maximum. There are no other extreme points, hence it is the only maximum on the line segment, and we have a supporting hyperplane that is also exposing the face  $F_{12}(s)$ .

FIGURE 11. Boundary of  $C$  identified with the unit sphere

*There are no other faces.* Observe that the faces  $F_1, F_2, F_{11}(s), F_{12}(s)$  and  $\{\{v\} : \gamma_1 \cup \gamma_2\}$  exhaust all possible faces of the set  $C$ , as glueing these sets together results in a topological sphere (see Fig. 11). We assume that the task of constructing an explicit homeomorphism between our faces and the surface of the two-sphere using this diagram is a triviality, and focus on showing that the existence of this homeomorphism guarantees that we have not missed any faces.

Observe that the set  $C$  is a compact convex set, hence there exists a sufficiently large ball centred in the interior of  $C$  such that  $C$  is a subset of the ball's interior. We can then construct a homeomorphism between the points on the boundary of  $C$  and the surface of this ball by considering the intersections between line segments connecting the centre and the points on the boundary of the ball and the boundary of  $C$ . For each such segment this intersection is unique (see [7, Remark 2.1.7]). We do not miss any boundary points that way because through each one we can draw a line segment connecting the centre of the ball with its boundary.

Having constructed this homeomorphism, we compose these two mappings that result in a homeomorphism between a sphere and its subset. This is impossible by the standard argument involving the stereographic projection and Borsuk-Ulam Theorem: if such homeomorphism existed, it is easy to construct another homeomorphism between the sphere and the Euclidean subspace of the same dimension by rotating the sphere and considering the stereographic projection. Being a homeomorphism, this is a continuous mapping, which by Borsuk-Ulam Theorem has to have coincident images of two antipodal points.

*Facial and tangential exposure.* We have already shown that one- and two-dimensional faces of  $C$  are exposed. The following simple statement guarantees that zero dimensional faces are exposed as well.

**Proposition 8.** *Suppose that  $E = F \cap G$ , where  $F$  and  $G$  are exposed faces of a closed convex set  $C \subset \mathbb{R}^n$ . Then  $E$  is an exposed face of  $C$ .*

*Proof.* Since both  $F$  and  $G$  are exposed, there exist  $p_F, p_G \in \mathbb{R}^n$  such that

$$\operatorname{Arg} \max_{x \in C} \langle p_F, x \rangle = F, \quad \operatorname{Arg} \max_{x \in C} \langle p_G, x \rangle = G.$$

Denote

$$m_F = \max_{x \in C} \langle p_F, x \rangle, \quad m_G = \max_{x \in C} \langle p_G, x \rangle.$$

Let  $p_E := p_F + p_G$ . We have

$$\begin{aligned} \langle p_E, x \rangle &= \langle p_F, x \rangle + \langle p_G, x \rangle < m_F + m_G \quad \forall x \in C \setminus (F \cap G); \\ \langle p_E, x \rangle &= \langle p_F, x \rangle + \langle p_G, x \rangle = m_F + m_G \quad \forall x \in E = F \cap G. \end{aligned}$$

Hence,

$$\operatorname{Arg\,max}_{x \in C} \langle p_E, x \rangle = E,$$

and therefore  $E$  is an exposed face of  $C$ .  $\square$

It is not difficult to observe (e.g. from the diagram in Fig. 11) that every zero-dimensional face of  $C$  is the intersection of at least two higher dimensional faces. Hence, all zero-dimensional faces are exposed.

The task of checking facial exposure can be reduced to verifying this property for two-dimensional faces only, as shown in the next proposition.

**Proposition 9.** *If a closed convex set  $C \subset \mathbb{R}^n$  is facially exposed, then all zero- and one-dimensional faces of  $C$  are tangentially exposed, i.e.*

$$(6) \quad \operatorname{span}(F - x) \cap \operatorname{Tangent}(x; C) = \operatorname{Tangent}(x; F) \quad \forall x \in F, \quad \forall F, \quad \dim F < 2.$$

*Proof.* Observe that all zero-dimensional faces are tangentially exposed due to the triviality of the relevant linear span, so we only need to prove the statement for one-dimensional faces.

Assume that there exists a face  $[u, v]$  of a closed facially exposed set  $C$  such that  $[u, v]$  is not tangentially exposed.

This means that there exists  $x \in [u, v]$  that violates (6). Observe that  $x \notin (u, v)$ , as for the points in the relative interior of the interval we have  $\operatorname{Tangent}(x; [u, v]) = \operatorname{span}(u - x)$ , and property (6) holds trivially. Without loss of generality we assume that  $x = u$ .

There exists a sequence  $(x_k)$  such that  $x_k \rightarrow u$ ,  $x_k \in C$ ,

$$p_k := \frac{x_k - u}{\|x_k - u\|} \rightarrow p \in (\operatorname{Tangent}(x; C) \cap \operatorname{span}\{v - u\}) \setminus \operatorname{Tangent}(u; F).$$

Observe that from  $p \notin \operatorname{Tangent}(u; F) = \operatorname{cone}\{v - u\}$ ,  $p \in \operatorname{span}\{v - u\}$ ,  $\|p\| = 1$  we deduce that

$$p = \frac{u - v}{\|u - v\|}.$$

Since  $\{u\}$  is an exposed face of  $C$ , there exists a normal  $q \in \mathbb{R}^n$  such that

$$\langle q, u \rangle > \langle q, x \rangle \quad \forall x \in C.$$

We therefore have

$$\langle q, p \rangle = \lim_{k \rightarrow \infty} \frac{\langle q, x_k - u \rangle}{\|x_k - u\|} \leq 0,$$

and on the other hand

$$\langle q, p \rangle = \frac{\langle q, u - v \rangle}{\|u - v\|} > 0,$$

a contradiction.  $\square$

It now remains to show tangential exposure for the two-dimensional faces, of which we only have two:  $F_1$  and  $F_2$ . Suppose that one of these faces, call it  $F$ , is not tangentially exposed. This implies that there exists  $x \in F$  and a sequence  $(x_k)$ ,  $x_k \rightarrow x$ ,  $x_k \in C$  such that

$$p_k = \frac{x_k - x}{\|x_k - x\|} \rightarrow p \in (\operatorname{Tangent}(x; C) \cap \operatorname{span}(F - x)) \setminus \operatorname{Tangent}(x; F).$$

Since  $p \in \operatorname{span}(F - x) \setminus \operatorname{Tangent}(x; F)$ , we have  $p \in \operatorname{Normal}(x; F) \cap \operatorname{span}(F - x)$ , and there must be a normal  $q \in \operatorname{Normal}(x; F) \cap \operatorname{span}(F - x)$  such that  $\langle p, q \rangle > 0$ .

This means that there is no normal  $h \in \operatorname{Normal}(x; C)$  such that

$$\Pi_{\operatorname{span} F}(h) = p,$$

otherwise for sufficiently large  $k$  we would have

$$\langle x_k - x, h \rangle < 0,$$



which is impossible.

We will show tangential exposure by demonstrating that each normal  $p \in \text{Normal}(x; F) \cap \text{span}(F - x)$  has a corresponding exposing normal  $h \in \text{Normal}(x; C)$  that projects onto  $p$  (when projected on the linear span of  $F$ ).

For the top face  $F_2 = \text{conv } \gamma_2$  the computation of the projections of normals is relatively straightforward. We first list all normals with corresponding faces to make sure that we do not miss anything (even though this is evident geometrically).

Since  $\text{span}(\text{conv } \gamma_2 - x) = \text{span}\{(1, 0, 0), (0, 1, 0)\}$  for all  $x \in F_2 = \text{conv } \gamma_2$ , we only need to consider the normals  $(v_1, v_2, 0) \in \mathbb{R}^2 \times \{0\}$ . We can therefore work in the two-dimensional space, so we let  $v = (v_1, v_2)$  and treat  $\gamma_2$  as a two-dimensional curve. We have

$$\max_{t \in [0, \varphi(1)]} \langle \gamma_2(t), v \rangle = \max_{u \in [0, \varphi(1)]} -tv_1 + t^2v_2.$$

By checking the necessary optimality condition for the expression on the right hand side, we obtain one local extremum  $t = v_1/2v_2$  for  $v_2 \neq 0$ , which is a maximum when  $v_2 < 0$ , and  $t \in [0, \varphi(1)]$  gives a further restriction  $v_1 < 0$  and  $v_1 \leq 2v_2\varphi(1)$ . For the other normals we compare the values at the endpoints to deduce

$$\max_{t \in [0, \varphi(1)]} = \begin{cases} -\frac{v_1^2}{4v_2^2} & v_1, v_2 < 0, v_1 \geq 2v_2\varphi(1) \quad (t^* = v_1/2v_2) \\ 0, & v_1 > \varphi(1)v_2, v_1 \geq 0 \quad (t^* = 0) \\ -\varphi(1)v_1 + \varphi(1)^2v_2, & v_1 < \varphi(1)v_2, v_1 \leq 2v_2\varphi(1) \quad (t^* = 0) \\ \text{conv}\{0, -\varphi(1)v_1 + \varphi(1)^2v_2\}, & v_1 = \varphi(1)v_2, v_1 > 0 \quad (t^* = 0, \varphi(1)). \end{cases}$$

Separating these cases by the support faces, we have for all points on the boundary

$$\text{Normal}(x, F_2) = \begin{cases} \text{cone}\{(-2t, -1)\}, & x = \gamma_1(t), \quad t \in (0, \varphi(1)), \\ \text{cone}\{(0, -1), (\varphi(1), 1)\}, & x = \gamma_1(0), \\ \text{cone}\{(\varphi(1), 1)\}, & x \in (\gamma_1(0)\gamma_1(1)), \\ \text{cone}\{(\varphi(1), 1), (-2\varphi_1(1), -1)\}, & x = \gamma_1(1). \end{cases}$$

We can verify explicitly that all these normals can be obtained as projections as the relevant points in  $\text{Normal}(x; K)$ . Indeed, for the points on the curve we have the normals exposing  $F_{12}(\varphi(s))$ ,

$$w(s) = \left( 2 \left( -\sqrt{7} - 1 \right) s^2, \left( \sqrt{7} - 5 \right) s, 4 \right).$$

These normals project onto

$$\text{cone} \left( 2 \left( -\sqrt{7} - 1 \right) \varphi(t), \left( \sqrt{7} - 5 \right) \right) = \text{cone} \left( 2 \left( \sqrt{7} - 5 \right) t, \left( \sqrt{7} - 5 \right) \right) = \text{cone}\{(-2t, -1)\},$$

as required.

Furthermore, for  $x \in \text{conv}[\gamma_2(0), \gamma_2(\varphi(1))]$  we have the normal to  $F_1$ ,

$$w := \left( 11 + 4\sqrt{7}, 3 \left( 2 + \sqrt{7} \right), -17 - 7\sqrt{7} \right).$$

Computing the projection we get

$$\text{cone}\left\{ \left( 11 + 4\sqrt{7}, 3(2 + \sqrt{7}) \right) \right\} = \text{cone}\left\{ \left( \frac{1}{3}(2 + \sqrt{7}), 1 \right) \right\} = \{(\varphi(1), 1)\}.$$

Convex hulls of these cones give the remaining normals for the ‘corners’.

For the remaining triangular face  $F_1$  observe that all its one-dimensional faces are exposed, hence the relevant normals project onto the normals at the points on these faces in the two-dimensional span of the face. The normals at the corner points are obtained as the convex hulls of these projections.

*Failure of strong tangential exposure.* To show that the second order tangential exposure is broken (and in fact the tangent cone is not even facially exposed), consider the tangent to the cone  $K$  at 0. We have

$$\text{Tangent}(0; K) = \limsup_{t \rightarrow \infty} tK = \text{cl cone}\{\gamma_1 \cup \gamma_2\}.$$

We scale our curves for convenience to obtain

$$\kappa_1(s) = (-1, -s, -s^2), \quad \kappa_2(t) = (-1, t, 0).$$

We hence have a slice of our tangent cone given by

$$\text{conv}\{(-s, -s^2), s \in [0, 1], (-1, t, 0), t \in [0, \varphi(1)]\},$$

see Fig. 7. It is clear that the set has an unexposed face  $\{(0, 0)\}$ .

*Failure of Facial Dual Completeness.* We will show that the cone  $K = \text{cone}\{C \times \{1\}\}$  is not FDC. For this we explicitly identify a parametrised family of points in the sum  $K^* + F^\perp$  whose limit does not belong to this set. Let

$$p(s) = \left(-2(\sqrt{7} + 1)s, (\sqrt{7} - 5), 0, -(\sqrt{7} + 3)s^2\right).$$

We will show that  $p(s) \in K^\circ + F^\perp$  for  $F = \text{cone}\{F_2 \times \{1\}\}$ , however,  $p(s) \rightarrow \bar{p} \notin K^\circ + F^\perp$ .

For the first relation, observe that  $F^\perp = \text{span}\{(0, 0, 1, 0)\}$ , and therefore

$$r(s) = (0, 0, -\frac{4}{s}, 0) \in F^\perp.$$

Hence,  $p(s) = q(s) + r(s)$ , where  $r(s) \in F^\perp$ , and we will next show that  $q(s) \in K^\circ$ .

We have explicitly

$$q(s) = \left(-2(\sqrt{7} + 1)s, (\sqrt{7} - 5), 4/s, -(\sqrt{7} + 3)s^2\right).$$

Abusing the notation and denoting by  $\gamma_1$  the lifted version of the relevant curve

$$\langle \gamma_1(u), q(s) \rangle = -(\sqrt{7} + 3 + 4\frac{u}{s})(u - s)^2 < 0$$

when  $u \neq s$ , also for  $\gamma_2$  substituting  $\varphi(u) = 1/3(2 + \sqrt{7})u$ ,

$$\langle \gamma_2(\varphi(u)), q(s) \rangle = -(3 + \sqrt{7})(u - s)^2,$$

which is less than zero unless  $u = s$ . We have hence shown that the point  $q(v)$  is in the polar.

Let

$$\bar{p} = \lim_{s \downarrow 0} p(s) = (0, \sqrt{7} - 5, 0, 0),$$

then

$$\langle \bar{p}, \gamma_1(s) \rangle = (5 - \sqrt{7})s > 0,$$

and hence  $\bar{p} \notin K^\circ$ .

**Technical computations for Example 3.** We begin with listing all exposed faces of  $C$  and then verifying that there are no other faces left. Thus we also list all faces of the cone  $K$  and verify that they are exposed.

*Faces of  $C$ .* Our goal is to list all faces of  $C = \text{conv}\{\gamma_1, \gamma_2\}$ . For the exposed faces

$$\mathcal{F} = \{\text{Arg max}_{x \in C} \langle x, v \rangle : v \in \mathbb{R}^n \setminus \{0\}\} = \{\text{conv Arg max}_{x \in \gamma_1 \cup \gamma_2} \langle x, v \rangle : v \in \mathbb{R}^n \setminus \{0\}\}.$$

Recall that every face of a convex set is either exposed, or is an exposed face of a larger face. Once we have listed all exposed proper faces, it will remain to check their sub-faces.

Fix a (normal) vector  $v \in \mathbb{R}^n$ ,  $v \neq 0$ . We have

$$\max_{x \in \gamma_1} \langle x, v \rangle = \max_{t \in [0, \pi/2]} v_1 \cos t + v_2 \sin t + v_3.$$

Solving the maximisation problem for  $t$ , we hence have

$$\max_{x \in \gamma_1} \langle x, v \rangle = \begin{cases} s + v_3, & v_2, v_1 > 0, \\ \max\{v_1, v_2\} + v_3, & \text{otherwise,} \end{cases}$$

where  $s = s(v_1, v_2) := \sqrt{v_1^2 + v_2^2}$ . Furthermore,

$$\text{Arg max}_{x \in \gamma_1} \langle x, v \rangle = \begin{cases} (v_1 s^{-1}, v_2 s^{-1}, 1), & v_1, v_2 > 0, \\ (0, 1, 1), & v_2 > v_1; v_1 \leq 0; \\ (1, 0, 1), & v_1 > v_2; v_2 \leq 0; \\ \{(1, 0, 1), (0, 1, 1)\}, & v_1 = v_2 < 0, \\ \gamma_1([0, \pi/2]), & v_1 = v_2 = 0. \end{cases}$$

Similar analysis for  $\gamma_2$  gives us

$$\max_{x \in \gamma_2} \langle x, v \rangle = \begin{cases} s - v_3, & v_2 > 0, \\ |v_1| - v_3, & \text{otherwise,} \end{cases}$$

and

$$\text{Arg max}_{x \in \gamma_2} \langle x, v \rangle = \begin{cases} (v_1 s^{-1}, v_2 s^{-1}, -1), & v_2 > 0, \\ (1, 0, -1), & v_2 \leq 0, v_1 > 0; \\ (-1, 0, -1), & v_2 \leq 0, v_1 < 0; \\ \{(1, 0, -1), (-1, 0, -1)\}, & v_2 < 0, v_1 = 0, \\ \gamma_2([0, \pi]), & v_2 = v_1 = 0. \end{cases}$$

We therefore have eight different regions with different families of maximal points over  $\gamma_1$  and  $\gamma_2$  (see the diagram in Fig. 12)

To determine whether we should include points from  $\gamma_1$ ,  $\gamma_2$  or from both components, we need to compare the maximal values of the support functions over each of these regions.

Region A:  $v_1, v_2 > 0$ . We have

$$\text{Arg max}_{x \in \gamma_1} \langle x, v \rangle = (v_1 s^{-1}, v_2 s^{-1}, 1), \quad \text{Arg max}_{x \in \gamma_2} \langle x, v \rangle = (v_1 s^{-1}, v_2 s^{-1}, -1),$$

$$\max_{x \in \gamma_1} \langle x, v \rangle = s + v_3, \quad \max_{x \in \gamma_2} \langle x, v \rangle = s - v_3,$$

$$\text{Arg max}_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle = \begin{cases} (v_1 s^{-1}, v_2 s^{-1}, 1), & v_3 > 0, \\ (v_1 s^{-1}, v_2 s^{-1}, -1), & v_3 < 0, \\ \{(v_1 s^{-1}, v_2 s^{-1}, 1), (v_1 s^{-1}, v_2 s^{-1}, -1)\}, & v_3 = 0. \end{cases}$$

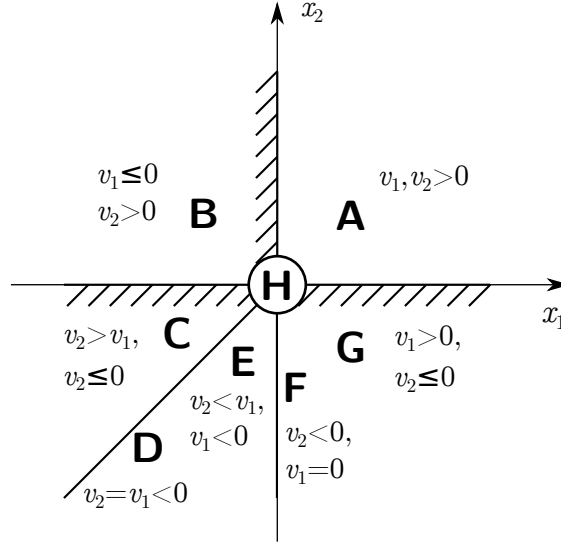


FIGURE 12. Eight regions for the families of normals

Region B:  $v_1 \leq 0, v_2 > 0$ . We have

$$\operatorname{Arg} \max_{x \in \gamma_1} \langle x, v \rangle = (0, 1, 1), \quad \operatorname{Arg} \max_{x \in \gamma_2} \langle x, v \rangle = (v_1 s^{-1}, v_2 s^{-1}, -1),$$

$$\max_{x \in \gamma_1} \langle x, v \rangle = \max\{v_1, v_2\} + v_3 = v_2 + v_3, \quad \max_{x \in \gamma_2} \langle x, v \rangle = s - v_3,$$

$$\operatorname{Arg} \max_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle = \begin{cases} (0, 1, 1), & 2v_3 > s - v_2, \\ (v_1 s^{-1}, v_2 s^{-1}, -1), & 2v_3 < s - v_2, \\ [(0, 1, 1), (v_1 s^{-1}, v_2 s^{-1}, -1)], & 2v_3 = s - v_2. \end{cases}$$

Region C:  $v_2 > v_1, v_1 < 0, v_2 \leq 0$ . We have

$$\operatorname{Arg} \max_{x \in \gamma_1} \langle x, v \rangle = (0, 1, 1), \quad \operatorname{Arg} \max_{x \in \gamma_2} \langle x, v \rangle = (-1, 0, -1),$$

$$\max_{x \in \gamma_1} \langle x, v \rangle = \max\{v_1, v_2\} + v_3 = v_2 + v_3, \quad \max_{x \in \gamma_2} \langle x, v \rangle = |v_1| - v_3 = -v_1 - v_3,$$

$$\operatorname{Arg} \max_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle = \begin{cases} (0, 1, 1), & 2v_3 > -v_1 - v_2, \\ (-1, 0, -1), & 2v_3 < -v_1 - v_2, \\ \{(0, 1, 1), (-1, 0, -1)\}, & 2v_3 = -v_1 - v_2. \end{cases}$$

Region D:  $v_1 = v_2 < 0$ . We have

$$\operatorname{Arg} \max_{x \in \gamma_1} \langle x, v \rangle = \{(1, 0, 1), (0, 1, 1)\}, \quad \operatorname{Arg} \max_{x \in \gamma_2} \langle x, v \rangle = (-1, 0, -1),$$

$$\max_{x \in \gamma_1} \langle x, v \rangle = \max\{v_1, v_2\} + v_3 = v_1 + v_3, \quad \max_{x \in \gamma_2} \langle x, v \rangle = |v_1| - v_3 = -v_1 - v_3,$$

$$\operatorname{Arg} \max_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle = \begin{cases} \{(1, 0, 1), (0, 1, 1)\}, & v_3 > -v_1, \\ (-1, 0, -1), & v_3 < -v_1, \\ \{(1, 0, 1), (0, 1, 1), (-1, 0, -1)\}, & v_3 = -v_1. \end{cases}$$

Region E:  $v_2 < v_1, v_2 < 0, v_1 < 0$ . We have

$$\begin{aligned} \operatorname{Arg max}_{x \in \gamma_1} \langle x, v \rangle &= (1, 0, 1), & \operatorname{Arg max}_{x \in \gamma_2} \langle x, v \rangle &= (-1, 0, -1), \\ \max_{x \in \gamma_1} \langle x, v \rangle &= \max\{v_1, v_2\} + v_3 = v_1 + v_3, & \max_{x \in \gamma_2} \langle x, v \rangle &= |v_1| - v_3 = -v_1 - v_3, \\ \operatorname{Arg max}_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle &= \begin{cases} (1, 0, 1), & v_3 > -v_1, \\ (-1, 0, -1), & v_3 < -v_1, \\ \{(1, 0, 1), (-1, 0, -1)\}, & v_3 = -v_1. \end{cases} \end{aligned}$$

Region F:  $v_1 = 0, v_2 < 0$ . We have

$$\begin{aligned} \operatorname{Arg max}_{x \in \gamma_1} \langle x, v \rangle &= (1, 0, 1), & \operatorname{Arg max}_{x \in \gamma_2} \langle x, v \rangle &= \{(1, 0, -1), (-1, 0, -1)\}, \\ \max_{x \in \gamma_1} \langle x, v \rangle &= \max\{v_1, v_2\} + v_3 = v_3, & \max_{x \in \gamma_2} \langle x, v \rangle &= |v_1| - v_3 = -v_3, \\ \operatorname{Arg max}_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle &= \begin{cases} (1, 0, 1), & v_3 > 0, \\ \{(1, 0, -1), (-1, 0, -1)\}, & v_3 < 0, \\ \{(1, 0, 1), (1, 0, -1), (-1, 0, -1)\}, & v_3 = 0. \end{cases} \end{aligned}$$

Region G:  $v_1 > 0, v_2 \leq 0$ . We have

$$\begin{aligned} \operatorname{Arg max}_{x \in \gamma_1} \langle x, v \rangle &= (1, 0, 1), & \operatorname{Arg max}_{x \in \gamma_2} \langle x, v \rangle &= (1, 0, -1), \\ \max_{x \in \gamma_1} \langle x, v \rangle &= \max\{v_1, v_2\} + v_3 = v_1 + v_3, & \max_{x \in \gamma_2} \langle x, v \rangle &= |v_1| - v_3 = v_1 - v_3, \\ \operatorname{Arg max}_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle &= \begin{cases} (1, 0, 1), & v_3 > 0, \\ (1, 0, -1), & v_3 < 0, \\ \{(1, 0, 1), (1, 0, -1)\}, & v_3 = 0. \end{cases} \end{aligned}$$

Region H:  $v_1 = v_2 = 0$ . We have

$$\begin{aligned} \operatorname{Arg max}_{x \in \gamma_1} \langle x, v \rangle &= \gamma_1([0, \pi/2]), & \operatorname{Arg max}_{x \in \gamma_2} \langle x, v \rangle &= \gamma_2([0, \pi]), \\ \max_{x \in \gamma_1} \langle x, v \rangle &= v_3 & \max_{x \in \gamma_2} \langle x, v \rangle &= -v_3, \\ \operatorname{Arg max}_{x \in \gamma_1 \cap \gamma_2} \langle x, v \rangle &= \begin{cases} \gamma_1([0, \pi/2]), & v_3 > 0, \\ \gamma_2([0, \pi]), & v_3 < 0, \\ \{\gamma_1([0, \pi/2]), \gamma_2([0, \pi])\}, & v_3 = 0. \end{cases} \end{aligned}$$

Notice that in the last line in the cases above is redundant, as we then have  $v_1 = v_2 = v_3 = 0$ , which is not considered.

Observe that for the regions A, B, C, E and G we have obtained one-dimensional exposed faces plus their zero dimensional exposed endpoints. There are no more nonempty subfaces of these exposed faces to explore.

For the region D we have a two dimensional triangular face with all subfaces exposed by either the remaining faces from region D or the one and two dimensional faces from regions C and E. All vertices are hence also exposed. Similarly for the region F the remaining two-dimensional faces come from the regions E and G.

The remaining region H generates two two-dimensional exposed faces that correspond to the convex hulls of the curves. All subfaces of these faces are clearly exposed, as they are also present as exposed faces in the other regions.

To compute the tangent cone at the point  $(0, 1, 1)$ , we can consider the intersection of all supporting hyperplanes to the set C that contain this point. It is not difficult to observe by checking the inclusion of this point in the relevant faces for each of the region that we have the following collection of normals coming from regions A, B, C and H:

$$\begin{aligned}
& \{(u, v, w), u \leq 0, v > 0, 2w \geq \sqrt{u^2 + v^2} - v\}, \\
& \{(u, v, w), v > u, u < 0, v \leq 0, 2w \geq -u - v\}, \\
& \{(u, u, w), u < 0, w \geq -u\}, \\
& \{(0, 0, w), w \geq 0\}.
\end{aligned}$$

Taking the closure of these sets won't change anything about the tangent cone which is a closed set. We can make these constraints more explicit as follows.

$$\begin{aligned}
\text{cl}\{(u, v, w) : u \leq 0, v \geq 0, 2w \geq \sqrt{u^2 + v^2} - v\} &= \text{cl cone}\{(u, 1, w) : u \leq 0, 2w \geq \sqrt{u^2 + 1} - 1\} \\
&= \text{cl cone}\{(u, 1, \frac{\sqrt{u^2 + 1} - 1}{2}), (0, 0, 1)\} : u \leq 0\} \\
&= \text{cl cone}(\{(2u, 2, \sqrt{u^2 + 1} - 1) : u \leq 0\} \cup \{(0, 0, 1)\})
\end{aligned}$$

$$\begin{aligned}
\text{cl}\{(u, v, w) : v \geq u, u \leq 0, v \leq 0, 2w \geq -u - v\} &= \text{cl}\{(-\lambda, \lambda v, \lambda w) : \lambda \geq 0, 0 \geq v \geq -1, 2w \geq 1 - v\} \\
&= \text{cl cone}\{(-1, v, w) : 0 \geq v \geq -1, 2w \geq 1 - v\} \\
&= \text{cl cone}\{(-1, v, (1 - v)/2), (0, 0, 1)\} : 0 \geq v \geq -1\} \\
&= \text{cone}\{(-1, -1, 1), (-1, 0, 1/2), (0, 0, 1)\};
\end{aligned}$$

$$\begin{aligned}
\text{cl}\{(u, v, w) : u = v, u \leq 0, w \geq -u\} &= \text{cl}\{(-\lambda, -\lambda, \lambda w) : \lambda \geq 0, w \geq 1\} \\
&= \text{cl cone}\{(-1, -1, w) : w \geq 1\} \\
&= \text{cone}\{(-1, -1, 1), (0, 0, 1)\};
\end{aligned}$$

$$\text{cl}\{(0, 0, w), w \geq 0\} = \text{cone}\{(0, 0, 1)\}.$$

Therefore, our tangent cone is defined by the normals

$$N = \text{cone}(\{(2u, 2, \sqrt{u^2 + 1} - 1) : u \leq 0\}, (0, 0, 1), (-1, 0, 1/2), (-1, -1, 1)).$$

The tangent cone is hence defined by the system

$$\begin{aligned}
z &\leq 0 \\
-x + \frac{1}{2}z &\leq 0 \\
-x - y + z &\leq 0 \\
2ux + 2y + (\sqrt{u^2 + 1} - 1)z &\leq 0 \quad \forall u \geq 0.
\end{aligned}$$

The last equation can be written as

$$z - 2y \geq \sup_{u \leq 0} (2ux + \sqrt{u^2 + 1}z).$$

Differentiating this in  $u$  we obtain the necessary condition for an internal maximum point on  $(-\infty, 0)$ ,

$$2x + \frac{u}{\sqrt{u^2 + 1}}z = 0,$$

which is only consistent when  $z$  and  $x$  have the same sign. It is also clear that  $|z| > |2x|$ . Solving for  $u$ , we have

$$u = \frac{2x}{\sqrt{z^2 - 4x^2}}.$$

Substituting the solution into the original function, we have

$$ux + \sqrt{u^2 + 1}z = \frac{4x^2}{\sqrt{z^2 - 4x^2}} + \frac{|z|z}{\sqrt{z^2 - 4x^2}} = -\sqrt{z^2 - 4x^2}.$$

For  $z < 2x \leq 0$  we have

$$-\sqrt{z^2 - 4x^2} > 2x \cdot 0 + \sqrt{0 + 1}z = z,$$

hence we have a maximum.

For  $2x \leq z$ ,  $z < 0$  we have

$$\lim_{u \rightarrow -\infty} ux + \sqrt{u^2 + 1}z = \lim_{u \rightarrow -\infty} \frac{u^2 4x^2 - (u^2 + 1)z^2}{2ux - \sqrt{u^2 + 1}z} = \lim_{u \rightarrow -\infty} \frac{u^2(4x^2 - z^2) - z^2}{2ux - \sqrt{u^2 + 1}z}.$$

This goes to  $+\infty$  as  $u \rightarrow -\infty$  for  $2x < z$ , hence, the expression is unbounded, and  $2x < z$  is impossible. For  $2x = z$  this goes to zero, and we hence have the supremum that equals  $-z$  (observe that for  $u = 0$  the expression is 0).

We have therefore

$$\sup_{u \leq 0} 2ux + \sqrt{u^2 + 1}z = \begin{cases} -\sqrt{z^2 - 4x^2}, & z \leq 2x \leq 0, \\ +\infty, & z \leq 0, 2x < z, \\ z, & z \leq 0, x \geq 0. \end{cases}$$

We can rewrite

$$z - 2y \geq \sup_{u \geq 0} (2ux + \sqrt{u^2 + 1}z)$$

as

$$\{2x \geq z\} \cap \left( \{z - 2y \geq -\sqrt{z^2 - 4x^2}, x \leq 0\} \cup \{z - 2y \geq z, x \geq 0\} \right).$$

Our final system is the union of two sets defined by the equations

$$\begin{aligned} z &\leq 0 \\ -x - y + z &\leq 0 \\ 2y - z &\leq \sqrt{z^2 - 4x^2} \\ 2x - z &\geq 0 \\ x &\leq 0 \end{aligned}$$

and

$$\begin{aligned} z &\leq 0 \\ -x - y + z &\leq 0 \\ y &\leq 0 \\ x &\geq 0 \\ 2x - z &\geq 0. \end{aligned}$$

Observe that the hyperplane  $y = 0$  supports this set with the outward normal  $(0, 1, 0)$ . This hyperplane exposes the face that we obtain by substituting  $y = 0$  in our equations:

$$\begin{aligned} z &\leq 0 \\ -x + z &\leq 0 \\ z &\leq \sqrt{z^2 - 4x^2} \\ 2x - z &\geq 0 \\ x &\leq 0 \end{aligned}$$

and

$$\begin{aligned} z &\leq 0 \\ -x + z &\leq 0 \\ x &\geq 0 \\ 2x - z &\geq 0. \end{aligned}$$

The first system yields  $x = 0$ ,  $z \leq 0$ , and the second system gives  $x \geq 0$ ,  $z \leq 0$  and  $2x \geq z$ .

It is not difficult to observe that  $x = 0, y = 0$  is a subface of this face. We will show that it is not exposed.

If there was an exposing hyperplane, then first of all we would have the normal perpendicular to  $(0, 0, 1)$ , so that we have  $p = (u, v, 0)$ . On the other hand, we need to ensure that for all points on the original larger face we have the strict inequality

$$\langle p, w \rangle < 0,$$

including the point  $(1, 0, -1)$ . So we have

$$u < 0.$$

The family of points  $(-\sqrt{t(t+1)}, -t, -1)$  satisfies the first system for  $t \downarrow 0$ , hence

$$-\sqrt{t(t+1)}u - tv < 0.$$

However,

$$\lim_{t \downarrow 0} -\sqrt{t(t+1)}u - tv = -u > 0,$$

a contradiction.

*Polar to*  $K = \text{cone}\{C \times \{1\}\}$ . Given the representation for our set  $C$  as

$$C = \{\langle p_t, \bar{x} \rangle \leq d_t, t \in T\},$$

it is not difficult to observe that its lifting is

$$K = \{\langle (p_t, -d_t), x \rangle \leq 0, t \in T\},$$

and its dual cone is therefore

$$K^* = \text{cl cone}\{(p_t, -d_t) \mid t \in T\}.$$

For each of the regions A through H we identify the relevant points in the dual cone. We also discard the points that do not contribute directly to the convex hull for a more compact representation.

Region A:  $v_1, v_2 > 0$ . We have

$$(p(v), d(v)) = \begin{cases} (v_1, v_2, v_3, -s - v_3), & v_3 \geq 0, \\ (v_1, v_2, v_3, -s + v_3), & v_3 < 0. \end{cases}$$

It is not difficult to observe that

$$\{(v_1, v_2, v_3, s + v_3) : v_1, v_2 > 0, v_3 \geq 0\} = \{(v_1, v_2, 0, -s) + v_3(0, 0, 1, -1) : v_1, v_2 > 0, v_3 \geq 0\}.$$

and

$$\{(v_1, v_2, v_3, -s + v_3) : v_1, v_2 > 0, v_3 < 0\} = \{(v_1, v_2, 0, -s) - v_3(0, 0, -1, -1) : v_1, v_2 > 0, -v_3 > 0\}.$$



Region B:  $v_1 \leq 0, v_2 > 0$ . We have

$$(p(v), d(v)) = \begin{cases} (v_1, v_2, v_3, -v_2 - v_3), & 2v_3 \geq s - v_2, \\ (v_1, v_2, v_3, -s + v_3), & 2v_3 < s - v_2, \end{cases}$$

We have

$$\begin{aligned} & \{(v_1, v_2, v_3, -v_2 - v_3) : v_1 \leq 0, v_2 > 0, 2v_3 \geq s - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(s - v_2), \frac{1}{2}(-s - v_2)) + \frac{1}{2}(2v_3 + v_2 - s)(0, 0, 1, -1) : v_1 \leq 0, v_2 > 0, 2v_3 \geq s - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(s - v_2), \frac{1}{2}(-s - v_2)) + \frac{1}{2}\lambda(0, 0, 1, -1) : v_1 \leq 0, v_2 > 0, \lambda \geq 0\}, \\ & \{(v_1, v_2, v_3, -s + v_3) : v_1 \leq 0, v_2 > 0, 2v_3 < s - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(s - v_2), \frac{1}{2}(-s - v_2)) - \frac{1}{2}(2v_3 + v_2 - s)(0, 0, -1, -1) : v_1 \leq 0, v_2 > 0, 2v_3 < s - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(s - v_2), \frac{1}{2}(-s - v_2)) + \frac{1}{2}\lambda(0, 0, -1, -1) : v_1 \leq 0, v_2 > 0, \lambda > 0\}. \end{aligned}$$

Region C:  $v_2 > v_1, v_1 < 0, v_2 \leq 0$ . We have

$$(p(v), d(v)) = \begin{cases} (v_1, v_2, v_3, -v_2 - v_3), & 2v_3 \geq -v_1 - v_2, \\ (v_1, v_2, v_3, v_1 + v_3), & 2v_3 < -v_1 - v_2, \end{cases}$$

We have

$$\begin{aligned} & \{(v_1, v_2, v_3, -v_2 - v_3) : v_2 > v_1, v_1 < 0, v_2 \leq 0, 2v_3 \geq -v_1 - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(-v_1 - v_2), \frac{1}{2}(-v_2 + v_1)) + \frac{1}{2}(2v_3 + v_1 + v_2)(0, 0, 1, -1) : v_2 > v_1, v_2 \leq 0, 2v_3 \geq -v_1 - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(-v_1 - v_2), \frac{1}{2}(-v_2 + v_1)) + \frac{1}{2}\lambda(0, 0, 1, -1) : v_2 - v_1 > 0, -v_2 \geq 0, \lambda \geq 0\} \\ &= \{(-1, 0, \frac{1}{2}, \frac{1}{2})(-v_2 + v_1) - (-1, -1, 1, 0)v_2 + \frac{1}{2}\lambda(0, 0, 1, -1) : v_2 - v_1 > 0, -v_2 \geq 0, \lambda \geq 0\} \\ &= \{\mu(-1, 0, \frac{1}{2}, -\frac{1}{2}) + \gamma(-1, -1, 1, 0) + \frac{1}{2}\lambda(0, 0, 1, -1) : \mu > 0, \gamma \geq 0, \lambda \geq 0\} \end{aligned}$$

$$\begin{aligned} & \{(v_1, v_2, v_3, v_1 + v_3) : v_2 > v_1, v_2 \leq 0, 2v_3 < -v_1 - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(-v_1 - v_2), \frac{1}{2}(-v_2 + v_1)) - \frac{1}{2}(2v_3 + v_2 + v_1)(0, 0, -1, 1) : v_2 > v_1, v_2 \leq 0, 2v_3 < -v_1 - v_2\} \\ &= \{(v_1, v_2, \frac{1}{2}(-v_1 - v_2), \frac{1}{2}(-v_2 + v_1)) + \frac{1}{2}\lambda(0, 0, -1, -1) : v_2 > v_1, v_2 \leq 0, \lambda > 0\} \\ &= \{\mu(-1, 0, \frac{1}{2}, -\frac{1}{2}) + \gamma(-1, -1, 1, 0) + \frac{1}{2}\lambda(0, 0, -1, -1) : \mu > 0, \gamma \geq 0, \lambda \geq 0\} \end{aligned}$$

Region D:  $v_1 = v_2 < 0$ . We have

$$(p(v), d(v)) = \begin{cases} (v_1, v_2, v_3, -v_1 - v_3), & v_3 \geq -v_1, \\ (v_1, v_2, v_3, v_1 + v_3), & v_3 < -v_1. \end{cases}$$

$$\begin{aligned}
& \{(v_1, v_2, v_3, -v_1 - v_3) : v_1 = v_2 < 0, v_3 \geq -v_1\} \\
& = \{(v_1, v_1, v_3, -v_1 - v_3) : -v_1 > 0, v_3 + v_1 \geq 0\} \\
& = \{-v_1(-1, -1, 1, 0) + (v_1 + v_3)(0, 0, 1, -1) : -v_1 > 0, v_3 + v_1 \geq 0\} \\
& = \{\lambda(-1, -1, 1, 0) + \mu(v_1 + v_3)(0, 0, 1, -1) : \lambda > 0, \mu \geq 0\}
\end{aligned}$$

$$\begin{aligned}
& \{(v_1, v_2, v_3, v_1 + v_3) : v_1 = v_2 < 0, v_3 < -v_1\} \\
& = \{(v_1, v_1, v_3, v_1 + v_3) : -v_1 > 0, -v_3 - v_1 > 0\} \\
& = \{-v_1(-1, -1, 1, 0) - (v_1 + v_3)(0, 0, -1, -1) : -v_1 > 0, -v_3 - v_1 > 0\} \\
& = \{\lambda(-1, -1, 1, 0) + \mu(v_1 + v_3)(0, 0, -1, -1) : \lambda > 0, \mu \geq 0\}
\end{aligned}$$

Region E:  $v_2 < v_1, v_2 < 0, v_1 < 0$ . We have

$$(p(v), d(v)) = \begin{cases} (v_1, v_2, v_3, -v_1 - v_3), & v_3 \geq -v_1, \\ (v_1, v_2, v_3, v_1 + v_3), & v_3 < -v_1. \end{cases}$$

$$\begin{aligned}
& \{(v_1, v_2, v_3, -v_1 - v_3) : v_2 < v_1, v_1 < 0, v_3 \geq -v_1\} \\
& = \{(v_1, v_2, v_3, -v_1 - v_3) : v_1 - v_2 > 0, -v_1 > 0, v_3 + v_1 \geq 0\} \\
& = \{-(-1, -1, 1, 0)v_1 + (0, -1, 0, 0)(v_1 - v_2) + (0, 0, 1, -1)(v_1 + v_3) : v_1 - v_2 > 0, -v_1 > 0, v_3 + v_1 \geq 0\} \\
& = \{(-1, -1, 1, 0)\lambda + (0, -1, 0, 0)\gamma + (0, 0, 1, -1)\mu : \lambda > 0, \gamma > 0, \mu \geq 0\}.
\end{aligned}$$

$$\begin{aligned}
& \{(v_1, v_2, v_3, v_1 + v_3) : v_2 < v_1, v_1 < 0, v_3 < -v_1\} \\
& = \{(v_1, v_2, v_3, v_1 + v_3) : v_1 - v_2 > 0, -v_1 > 0, -v_3 - v_1 > 0\} \\
& = \{-(-1, -1, 1, 0)v_1 + (0, -1, 0, 0)(v_1 - v_2) + (0, 0, -1, -1)(-v_1 - v_3) : v_1 - v_2 > 0, -v_1 > 0, v_3 + v_1 \geq 0\} \\
& = \{(-1, -1, 1, 0)\lambda + (0, -1, 0, 0)\gamma + (0, 0, -1, -1)\mu : \lambda > 0, \gamma > 0, \mu > 0\}.
\end{aligned}$$

Region F:  $v_1 = 0, v_2 < 0$ . We have

$$(p(v), d(v)) = \begin{cases} (v_1, v_2, v_3, -v_3), & v_3 \geq 0, \\ (v_1, v_2, v_3, v_3), & v_3 < 0. \end{cases}$$

$$\begin{aligned}
& \{(0, v_2, v_3, -v_3) : -v_2 > 0, v_3 \geq 0\} \\
& = \{-v_2(0, -1, 0, 0) + (0, 0, 1, -1)v_3 : -v_2 > 0, v_3 \geq 0\} \\
& = \{(0, -1, 0, 0)\lambda + (0, 0, 1, -1)\mu : \lambda > 0, \mu \geq 0\}
\end{aligned}$$

$$\begin{aligned}
& \{(0, v_2, v_3, v_3) : -v_2 > 0, -v_3 \geq 0\} \\
& = \{-v_2(0, -1, 0, 0) - (0, 0, -1, -1)v_3 : -v_2 > 0, -v_3 \geq 0\} \\
& = \{(0, -1, 0, 0)\lambda + (0, 0, -1, -1)\mu : \lambda > 0, \mu > 0\}
\end{aligned}$$

Region G:  $v_1 > 0, v_2 \leq 0$ . We have

$$(p(v), d(v)) = \begin{cases} (v_1, v_2, v_3, -v_1 - v_3), & v_3 \geq 0, \\ (v_1, v_2, v_3, -v_1 + v_3), & v_3 < 0. \end{cases}$$

$$\begin{aligned}
& \{(v_1, v_2, v_3, -v_1 - v_3) : v_1 > 0, v_2 \leq 0, v_3 \geq 0\} \\
&= \{(1, 0, 0, 1)v_1 - (0, -1, 0, 0)v_2 + (0, 0, 1, -1)v_3 : v_1 > 0, -v_2 \geq 0, v_3 \geq 0\} \\
&= \{(1, 0, 0, 1)\mu - (0, -1, 0, 0)\gamma + (0, 0, 1, -1)\lambda : \mu > 0, \gamma \geq 0, \lambda \geq 0\} \\
& \\
& \{(v_1, v_2, v_3, -v_1 + v_3) : v_1 > 0, v_2 \leq 0, v_3 < 0\} \\
&= \{(1, 0, 0, -1)v_1 - (0, -1, 0, 0)v_2 - (0, 0, -1, -1)v_3 : v_1 > 0, -v_2 \geq 0, -v_3 > 0\} \\
&= \{(1, 0, 0, -1)\mu - (0, -1, 0, 0)\gamma + (0, 0, -1, -1)\lambda : \mu > 0, \gamma \geq 0, \lambda > 0\}
\end{aligned}$$

Region H:  $v_1 = v_2 = 0$ . We have

$$(p(v), d(v)) = \begin{cases} (0, 0, v_3, -v_3), & v_3 > 0, \\ (0, 0, v_3, v_3), & v_3 < 0. \end{cases}$$

$$\begin{aligned}
& \{(0, 0, v_3, -v_3) : v_3 > 0\} \\
&= \{(0, 0, 1, -1)\mu : \mu > 0\}
\end{aligned}$$

$$\begin{aligned}
& \{(0, 0, v_3, v_3) : -v_3 < 0\} \\
&= \{(0, 0, -1, -1)\mu : \mu > 0\}
\end{aligned}$$

Taking the conic hull and closure and discarding repeated points we obtain our polar cone

$$\begin{aligned}
K^\circ &= \text{cl cone}\{(t_1, t_2, 0, -\sqrt{t_1^2 + t_2^2}) : t_1, t_2 > 0\}, \\
& \left\{ (2t_3, 2t_4, \sqrt{t_3^2 + t_4^2} - t_4, -\sqrt{t_3^2 + t_4^2} - t_4), t_3 \leq 0, t_4 > 0 \right\}, \\
& (0, 0, 1, -1), (0, 0, -1, -1), (-2, 0, 1, -1), (-1, -1, 1, 0), (0, -1, 0, 0), (0, 0, 0, -1), (1, 0, 0, -1)\} \\
&= \text{cone}\{(t_1, t_2, 0, -\sqrt{t_1^2 + t_2^2}) : t_1, t_2 \geq 0\}, \\
& \left\{ (2t_3, 2t_4, \sqrt{t_3^2 + t_4^2} - t_4, -\sqrt{t_3^2 + t_4^2} - t_4), t_3 \leq 0, t_4 \geq 0 \right\}, \\
& (0, 0, 1, -1), (0, 0, -1, -1), (-1, -1, 1, 0), (0, -1, 0, 0), (0, 0, 0, -1)\} \\
&= \text{cone}\{(\cos t, \sin t, 0, -1) : t \in [0, \pi/2]\}, \\
& \left\{ (\cos \tau, \sin \tau, \frac{1 - \sin \tau}{2}, \frac{-1 - \sin \tau}{2}), \tau \in [\pi/2, \pi] \right\}, \\
& (0, 0, 1, -1), (0, 0, -1, -1), (-1, -1, 1, 0), (0, -1, 0, 0), (0, 0, 0, -1)\}.
\end{aligned}$$

This is because zero is not in the convex hull of the set inside the brackets. We can verify this by computing inner products with an interior point of the primal cone, say,  $p = (0, 1, 0, 2)$ .

*Facial Dual Completeness of  $K$ .* To check whether  $K$  is facially dual complete, it remains to consider all possible sums  $F^\perp + K^\circ$  for orthogonal complements of faces of  $K$  and see if this set is closed.

Notice that whenever the face  $F$  is one dimensional, its orthogonal complement is a three dimensional subspace. Its sum with any closed cone is closed, since the relevant one-dimensional projection of a closed cone is closed. This also works for two dimensional faces, as proved in the next proposition.

**Proposition 10.** *Let  $K \subseteq \mathbb{R}^n$ , and assume that  $K$  is facially exposed. Then for every  $F \triangleleft K$  such that  $F = \text{cone}\{p_1, p_2\}$ , where  $p_1, p_2 \in \mathbb{R}^n$  are linearly independent, the set  $K^* + F^\perp$  is closed.*

*Proof.* Since  $K$  is facially exposed, the faces  $E_1 = F \cap \text{span } p_1$  and  $E_2 = F \cap \text{span } p_2$  are exposed. Therefore, there are  $h_1, h_2 \in \mathbb{R}^n$  such that

$$(7) \quad \langle h_i, p_i \rangle = 0, \quad \langle h_1, x \rangle < 0 \quad \forall x \in K \setminus E_i, \quad i \in \{1, 2\}.$$

Observe that  $h_1, h_2 \notin F^\perp$  (since they expose proper faces of  $F$ ). Hence,

$$g_i := \Pi_{\text{span } F}(h_i) \neq 0 \quad \forall i \in \{1, 2\}.$$

Moreover,

$$\langle g_i, p_i \rangle = \langle g_i - h_1, p_i \rangle + \langle h_1, p_i \rangle = 0 \quad \forall i \in \{1, 2\},$$

since  $g_i - h_i \in F^\perp$ , and likewise

$$\langle g_i, x \rangle < 0 \quad \forall x \in F \setminus E_i, \quad i \in \{1, 2\}.$$

Observe that any  $x \in \text{span } F$  can be written as

$$x = \alpha p_1 + \beta p_2, \quad \alpha, \beta \in \mathbb{R},$$

with  $\alpha, \beta \geq 0$  if and only if  $x \in F$ .

We have

$$\langle x, g_1 \rangle = \alpha \langle p_1, g_1 \rangle + \beta \langle p_2, g_1 \rangle = \beta \langle p_2, g_1 \rangle,$$

and

$$\langle x, g_2 \rangle = \alpha \langle p_1, g_2 \rangle.$$

It follows from these relations that  $\alpha \geq 0$  if and only if  $\langle x, g_1 \rangle \leq 0$  and  $\beta \geq 0$  if and only if  $\langle x, g_2 \rangle \leq 0$ . We have the representation

$$F = \{x \in \mathbb{R}^n : \langle x, g_1 \rangle \leq 0, \langle x, g_2 \rangle \leq 0\} \cap \text{span } F.$$

For the polar we have

$$F^\circ = \text{cl cone}\{g_1, g_2\} + F^\perp = \text{cone}\{g_1, g_2\} + F^\perp.$$

Finally, observe that for any  $y \in F^\circ$  we have

$$y = \alpha g_1 + \beta g_2 + u,$$

where  $\alpha, \beta \in \mathbb{R}_+$  and  $u \in F^\perp$ . We can rewrite this as

$$y = \alpha g_1 + \beta g_2 + u = \alpha h_1 + \beta h_2 + (\alpha(g_1 - h_1) + \beta(g_2 - h_2) + u),$$

where  $\alpha(g_1 - h_1) + \beta(g_2 - h_2) + u \in F^\perp$ , and since  $h_1, h_2 \in K^\circ$ , we have  $y \in K^\circ + F^\perp$ . By the arbitrariness of  $y$  this yields  $F^\circ \subset K^\circ + F^\perp$ . Together with  $F^\circ = \text{cl } K^\circ + F^\perp$  this yields  $K^\circ + F^\perp = \text{cl } K^\circ + F^\perp$ .

□

Due to our observation about one-dimensional faces and Proposition 10 to prove that the cone  $K = \text{cone}\{C \times \{1\}\}$  is FDC we only need to check the closedness of  $F^\perp + K^*$  for the three dimensional faces of  $K$  (that correspond to the two dimensional faces of  $C$  shown in Fig 13).

Checking whether  $F^\perp + K^\circ$  is closed is the same as checking whether the projection of  $K^\circ$  on  $\text{span } F$  is closed.

Moreover, if  $K = \text{cone } S$ , where  $S \subset \mathbb{R}^n$  is a compact set such that  $0 \notin \text{conv } S$ , then

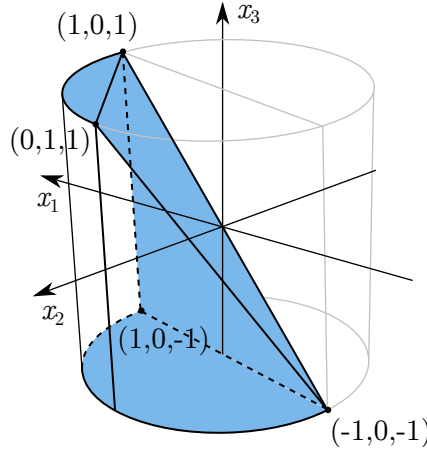
$$\Pi_L(K) = \text{cone } \Pi_L(S)$$

for any linear subspace  $L$ .

To compute the projections we use a coordinate transformation that rotates the space so that  $F^\perp$  coincides with the last coordinate. This allows us to obtain a graphic representation of the set  $S$  for each case.

For every three dimensional face  $F \triangleleft K$  we apply a unitary transformation  $U$  to our coordinate system such that

$$U(\Pi_{\text{span } F^\perp}(x)) \in \mathbb{R}^3 \times \{0\}.$$

FIGURE 13. Two dimensional faces of  $C$ 

Let  $p_1, p_2, p_3$  be an orthonormal basis of  $F$  and let  $F^\perp = \text{span } p_4$ ,  $\|p_4\| = 1$ . Then

$$U_i(x) = \langle p_i, x \rangle.$$

We will use the representation  $K = \text{cone } S$  where

$$S = S_1 \cup S_2 \cup S_3,$$

$$S_1 = \{(\cos t, \sin t, 0, -1) : t \in [0, \pi/2]\},$$

$$S_2 = \left\{ \left( \cos \tau, \sin \tau, \frac{1 - \sin \tau}{2}, \frac{-1 - \sin \tau}{2} \right), \tau \in [\pi/2, \pi] \right\},$$

$$S_3 = \{(0, 0, 1, -1), (0, 0, -1, -1), (-1, -1, 1, 0), (0, -1, 0, 0), (0, 0, 0, -1)\}.$$

For the top face we have  $F = \text{cone}\{\gamma_1 \times \{1\}\}$ , and the relevant linear span is

$$\text{span } F = \text{span}\{(1, 0, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1)\}.$$

The orthogonal complement is

$$F^\perp = \text{span}(0, 0, 1 - 1).$$

We first find the orthonormal basis  $p_1, p_2, p_3, p_4$ :

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1/\sqrt{2}, 1/\sqrt{2}), (0, 0, 1/\sqrt{2}, -1/\sqrt{2}).$$

We have

$$U(S_1) = \left\{ \{(\cos t, \sin t, -1/\sqrt{2}, 1/\sqrt{2}) : t \in [0, \pi/2]\} \right\},$$

$$U(S_2) = \left\{ (\cos \tau, \sin \tau, -1/\sqrt{2} \sin \tau, 1/\sqrt{2}), \tau \in [\pi/2, \pi] \right\},$$

$$U(S_3) = \left\{ (0, 0, 0, \sqrt{2}), (0, 0, -\sqrt{2}, 0), (-1, -1, 1/\sqrt{2}, 1/\sqrt{2}), (0, -1, 0, 0), (0, 0, -1/\sqrt{2}, 1/\sqrt{2}) \right\}.$$

Our projection is therefore a three dimensional set  $P = \text{cone } S'$ , where  $S' = \{S'_1, S'_2, S'_3\}$ ,

$$S'_1 = \left\{ \{(\cos t, \sin t, -1/\sqrt{2}) : t \in [0, \pi/2]\} \right\},$$

$$S'_2 = \left\{ (\cos \tau, \sin \tau, -1/\sqrt{2} \sin \tau), \tau \in [\pi/2, \pi] \right\},$$

$$S'_3 = \left\{ (0, 0, 0), (0, 0, -\sqrt{2}), (-1, -1, 1/\sqrt{2}), (0, -1, 0), (0, 0, -1/\sqrt{2}) \right\}.$$

Let  $w = (1, 1, z)$ , where  $z \in (2, 2\sqrt{2})$ . We will show that

$$\langle w, x \rangle < 0$$

for all points in  $S'$  except for 0. This will effectively show that the relevant cone is closed.

We have for points in  $S'_3 \setminus (0, 0, 0)$

$$\langle w, S'_3 \setminus (0, 0, 0) \rangle = \{-z\sqrt{2}, -2 + z/\sqrt{2}, -1, -z/\sqrt{2}\} < 0;$$

$$\langle w, S'_1 \rangle = \{\cos t + \sin t - z/\sqrt{2}, t \in [0, \pi/2]\}$$

$$\langle w, S'_2 \rangle = \{\cos t + \sin t - z/\sqrt{2} \sin t, t \in [\pi/2, \pi]\}$$

It is evident that all numbers on the right hand side are negative.

For the bottom face we have  $F = \text{cone}\{\gamma_1 \times \{1\}\}$ , and the relevant linear span is

$$\text{span } F = \text{span}\{(1, 0, 1, -1), (0, 1, 1, -1), (0, 0, 1, -1)\}.$$

The orthogonal complement is

$$F^\perp = \text{span}(0, 0, 1, 1).$$

We find the orthonormal basis  $p_1, p_2, p_3, p_4$ :

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1/\sqrt{2}, -1/\sqrt{2}), (0, 0, 1/\sqrt{2}, 1/\sqrt{2}).$$

We have

$$U(S_1) = \left\{ \{(\cos t, \sin t, 1/\sqrt{2}, -1/\sqrt{2}) : t \in [0, \pi/2]\} \right\},$$

$$U(S_2) = \left\{ (\cos \tau, \sin \tau, 1/\sqrt{2}, -1/\sqrt{2} \sin \tau), \tau \in [\pi/2, \pi] \right\},$$

$$U(S_3) = \left\{ (0, 0, \sqrt{2}, 0), (0, 0, 0, -\sqrt{2}), (-1, -1, 1/\sqrt{2}, 1/\sqrt{2}), (0, -1, 0, 0), (0, 0, 1/\sqrt{2}, -1/\sqrt{2}) \right\}.$$

Our projection is therefore a three dimensional set  $P = \text{cone } S'$ , where  $S' = \{S'_1, S'_2, S'_3\}$ ,

$$S'_1 = \left\{ \{(\cos t, \sin t, 1/\sqrt{2}) : t \in [0, \pi/2]\} \right\},$$

$$S'_2 = \left\{ (\cos \tau, \sin \tau, 1/\sqrt{2}), \tau \in [\pi/2, \pi] \right\},$$

$$S'_3 = \left\{ (0, 0, \sqrt{2}), (0, 0, 0), (-1, -1, 1/\sqrt{2}), (0, -1, 0), (0, 0, 1/\sqrt{2}) \right\}.$$

Let  $w = (0, y, -1)$ , where  $y \in (0, 1/\sqrt{2})$ . We will show that

$$\langle w, x \rangle < 0$$

for all points in  $S'$  except for 0. This will effectively show that the relevant cone is closed.

We have for points in  $S'_3 \setminus (0, 0, 0)$

$$\langle w, S'_3 \setminus (0, 0, 0) \rangle = \{-\sqrt{2}, -y - 1/\sqrt{2}, -y, -1/\sqrt{2}\} < 0;$$

$$\langle w, S'_1 \rangle = \{y \sin t - 1/\sqrt{2}, t \in [0, \pi/2]\}$$

$$\langle w, S'_2 \rangle = \{y \sin t - 1/\sqrt{2}, t \in [\pi/2, \pi]\}$$

It is evident that all numbers on the right hand side are negative.

The remaining triangular faces satisfy the Pataki criterion (see [15, Theorem 3]): if  $K$  is facially exposed and all properly minimal faces of  $F^*$  are exposed, then  $K$  is facially dual complete. Since the triangular faces are polyhedral, their duals are also polyhedral, and have all their faces exposed.